ALGEBRAICAL PROBLEMS,

PRODUCING

SIMPLE *and* QUADRATIC EQUATIONS,

WITH THEIR SOLUTIONS.

DESIGNED AS

AN INTRODUCTION TO THE HIGHER BRANCHES OF ANALYTICS.

BY THE

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ADVERTISEMENT.

THE following pages contain a collection of Algebraical Problems designed solely to point out the various methods employed by Analysts in the Solution of Equations. They are arranged in the usual manner: 1. Simple Equations; 2. Pure Quadratics, and others which may be solved without completing the square; and 3. Affected Quadratics. Many books have been consulted; and, as utility is the sole object of this Publication, wherever a proper Example occured, it has been taken without hesitation, or altered to suit the purpose. At the head of each Section are given the common Rules; and the whole concludes with a Collection of Problems without Solutions, for the Exercise of the Learner.

ERRATA.

P. 112, line 6. for $\sqrt{y+2}$, read $\sqrt{y}+2$.

P. 208, line 21. for -42, read -52.

ALGEBRAICAL PROBLEMS.

SECT. I.

DEFINITIONS.

(1.) An Equation is a proposition, which declares the equality of two quantities, expressed algebraically.

This is done by connecting these quantities, by the sign (=); thus, $x-4=6-x$ is an equation expressing the equality of the quantities $x-4$ and $6-x$. Also $x-5=0$ is an equation which asserts that $x-5$ is equal to nothing, and therefore that the positive part of the expression is equal to the negative part.

(2.) A Simple Equation is one which, when cleared of fractions and surds, contains only the first power of the unknown quantity.

(3.) A Quadratic Equation, or an equation of two dimensions, is one into which the square of the unknown quantity enters, with or without the simple power.

(4.) A Cubic Equation, or an equation of three dimensions, is one into which the cube of the unknown quantity enters, with or without the simple and quadratic powers.

(5.) In general, the index of the highest power of the unknown quantity denotes the number of dimensions of the equation.

(6.) A Pure Quadratic is one into which only the square of the unknown quantity enters.

(7.) An Adfected Quadratic is one which involves the square of the unknown quantity, and also the simple power and known quantities.

Thus, $ax^2+b=0$ is a pure quadratic,

and $ax^2+bx+c=0$ is an adfected quadratic.

(8.) The Resolution of Equations is the determining, from some quantities given, the values of others which are unknown, so that these latter may answer certain conditions proposed.

(9.) And these values are called Roots of the Equation.

(10.) Known quantities are usually expressed by the first letters of the alphabet, *a*, *b*, *c*, &c.; and unknown quantities by the last, *v*, *x*, *y*, &c. And this must be always understood, unless the contrary be expressed.

AXIOMS.

(11.) If equal quantities be added to equal quantities, the sums will be equal.

(12.) If equal quantities be taken from equal quantities, the remainders will be equal.

(13.) If equal quantities be multiplied by the same or equal quantities, the products will be equal.

(14.) If equal quantities be divided by the same or equal quantities, the quotients will be equal.

(15.) If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

(16.) If a quantity be both multiplied and divided by another, its value will not be altered.

(17.) Any quantity may be transposed from one side of an equation to the other, by changing its sign:

Because, in this transposition, the same quantity is merely subtracted from each side of the equation; and (12) if equals be taken from equals, the remainders are equal.

Thus, $5x-7=2x+2$, if $2x$ be taken from each side, $5x-2x-7=2$, or $3x-7=2$; and if -7 be subtracted, or (which is the same thing) if $+7$ be added to each side, $3x=2+7=9$.

Also, if $x-a+b=c-3x$, then, by subtracting $-a+b-3x$ from each side, we have $x+3x=a-b+c$.

Cor. 1. Hence, if the signs of all the terms on each side of an equation be changed, the two sides still remain equal; because in this change every term is transposed.

Cor. 2. Hence, when the known and unknown quantities are connected in an equation by the signs + or −, they may be separated by transposing the known quantities to one side, and the unknown to the other.

Cor. 3. Hence also, if any quantity be found on both sides of an equation, it may be taken away from each; thus, if $x+y=5+y$, then $x=5$. If $a-b=c+d-b$, then $a=c+d$.

(18.) If every term on each side of an equation be multiplied by the same quantity, the results will be equal:

Because in multiplying every term on each side by any quantity, the value of the whole side is multiplied by that quantity; and (13) if equals be multiplied by the same quantity, the products will be equal.

Thus, if $x=5+a$, then $6x=30+6a$, by multiplying every term by 6.

COR. 1. Hence an equation, of which any part is fractional, may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. If there be more fractions than one in the given equation, it may be so reduced by multiplying every term either by the product of the denominators, or by a common multiple of them; and if the least common multiple be used, the equation will be in its lowest terms.

Thus, if $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13$; if every term be multiplied by 12, which is the least common multiple of 2, 3, 4; $6x+4x+3x=156$.

COR. 2. Hence also, if every term on both sides have a common *multiplier* or *divisor*, that common multiplier or divisor may be taken away;

Thus, if $\frac{5x}{4}+\frac{a+6}{4}=\frac{4x-7}{4}$, then also $5x+a+6=4x-7$;

Also, if $\frac{ax+ab}{c}=\frac{ad}{c}+\frac{4ax}{c}$, then, multiplying by $\frac{c}{a}$, $x+b=d+4x$.

(19.) If each side of an equation be raised to the same power, the results are equal;

Thus, if $x=6$, $x^2=36$; if $x+a=y-b$, then $x^2+2ax+a^2=y^2-2by+b^2$;

And if the same roots be extracted on each side, the results are equal:

Thus, if $x^2=49$, $x=7$; if $x^2+2x+1=y^2-y+\frac{1}{4}$, then $x+1=y-\frac{1}{2}$;

For (13 and 14) when equal quantities on each side of an equation are multiplied or divided by equal quantities, the results will be equal.

Cor. Hence, if that side of the equation which contains the unknown quantity be a perfect square, cube, or other power, by extracting the square root, cube root, &c. of both sides, the equation will be reduced to one of lower dimensions:

Thus, if $x^2+8x+16=36$, $x+4=6$,
if $x^3+3x^2+3x+1=27$, $x+1=3$,
if $x^4+2x^3+x^2=100$, $x^2+x=10$.

(20.) Any equation may be cleared of a single radical quantity by transposing all the other terms to the contrary side, and raising each side to the power denominated by the surd. If there are more than one surd, the operation must be repeated.

Thus, if $\sqrt{x^2+7}+x=7$,
then (17) by transposition, $\sqrt{x^2+7}=7-x$;
and (19) by squaring each side, $x^2+7=49-14x+x^2$, which is free from surds.

Also, if $\sqrt{x^2+\sqrt{x^2+21}}-1=x$,
then (17) by transposition, $\sqrt{x^2+\sqrt{x^2+21}}=x+1$,
and (19) by squaring each side, $x^2+\sqrt{x^2+21}=x^2+2x+1$;
$\therefore$ (17. Cor. 3.) $\sqrt{x^2+21}=2x+1$,
and (19) by squaring each side, $x^2+21=4x^2+4x+1$, which is free from surds.

(21.) Any proportion may be converted into an equation; for the product of the extremes is equal to the product of the means.

Let $a:b::c:d$, by the nature of proportion $\frac{a}{b}=\frac{c}{d}$;
$\therefore$ (Cor. 18.) $ad=bc$.

(22.) *Examples in which the preceding Rules are applied, in the Solution of Equations.*

1. Given $4x+36=5x+34$, to find the value of x.

(17) By transposition, $36-34=5x-4x$,
and therefore $2=x$.

2. Given $x-7=\frac{x}{5}+\frac{x}{3}$, to find the value of x.

Here 15, the product of 3 and 5, being their least common multiple, every term must be multiplied by it (18. Cor. 1.), and $15x-105=3x+5x$;

$\therefore$ (17) by transposition, $15x-3x-5x=105$,
or $7x=105$;

and therefore (18. Cor. 2.) $x=\frac{105}{7}=15$.

3. Given $3ax-4ab=2ax-6ac$, to find the value of x in terms of b and c.

(18. Cor. 2.) dividing every term by a, $3x-4b=2x-6c$;
$\therefore$ (17) by transposition, $3x-2x=4b-6c$,
or $x=4b-6c$.

4. Given $3x^2-10x=8x+x^2$, to find the value of x.

(18. Cor. 2.) dividing every term by x, $3x-10=8+x$;
$\therefore$ (17) by transposition, $3x-x=8+10$,
or $2x=18$;

$\therefore$ (18. Cor. 2.) $x=\frac{18}{2}=9$.

5. Given $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$, to find the value of x.

Here 12 is the least common multiple of 2, 3, and 4; (18. Cor. 1.) multiplying both sides of the equation therefore by 12, $6x+4x=3x+84$;

$\therefore$ (17) by transposition, $6x+4x-3x=84$,

or $7x=84$;

$\therefore$ (18. Cor. 2.) $x=\frac{84}{7}=12$.

6. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$, to find the value of x.

(18. Cor. 1.) multiplying by 20, the least common multiple of 4 and 5,

$5x-25+120x=1136-4x$;

$\therefore$ (17) by transposition, $5x+120x+4x=1136+25$,

or $129x=1161$;

$\therefore$ (18. Cor. 2.) $x=\frac{1161}{129}=9$.

7. Given $x+\frac{11-x}{3}=\frac{19-x}{2}$, to find the value of x.

(18. Cor. 1.) multiplying by 6, the least common multiple of 2 and 3,

$6x+22-2x=57-3x$;

$\therefore$ (17) by transposition, $6x-2x+3x=57-22$,

or $7x=35$;

$\therefore$ (18. Cor. 2.) $x=\frac{35}{7}=5$.

8. Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$, to find the value of x.

(18. Cor. 1.) multiplying by 10, the least common multiple of 2 and 5,

$$30x+4x+12=50+55x-185;$$

$\therefore$ (17) by transposition, $12-50+185=55x-30x-4x$,

or $147=21x$;

$\therefore$ (18. Cor. 2.) $\frac{147}{21}=7=x$.

9. Given $\frac{6x-4}{3}-2=\frac{18-4x}{3}+x$, to find the value of x.

(18. Cor. 1.) Multiplying every term by 3,

$$6x-4-6=18-4x+3x;$$

and $\therefore$ (17) by transposition, $6x+4x-3x=18+6+4$,

or $7x=28$;

$\therefore$ (18. Cor. 2.) $x=\frac{28}{7}=4$.

10. Given $21+\frac{3x-11}{16}=\frac{5x-5}{8}+\frac{97-7x}{2}$, to find the value of x.

Since 16 contains 8 and 2, a certain number of times exactly, it will be the least common multiple of 16, 8, and 2; and therefore (18. Cor. 1.) multiplying both sides of equation by 16,

$$336+3x-11=10x-10+776-56x;$$

$\therefore$ (17) by trans^n, $3x-10x+56x=11-10+776-336$,

or $49x=441$;

$\therefore$ (18. Cor. 2.) $x=\frac{441}{49}=9$.

11. Given $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$; to find the value of x.

(18. Cor. 1.) multiplying both sides of equation by 6, the product of 2 and 3,

$$6x+9x-15=72-4x+8;$$

$\therefore$ (17) by transposition, $6x+9x+4x=72+8+15$,

or $19x=95$;

$\therefore$ (18. Cor. 2.) $x=\frac{95}{19}=5$.

12. Given $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}$, to find the value of x.

Since 12 is a multiple of 3 and 4, it is the least common multiple of 3, 4, and 12; therefore (18. Cor. 1.) multiplying both sides of the equation by 12,

$$36x-3x+12-48=20x+56-1;$$

$\therefore$ (17) by trans^n, $36x-3x-20x=56+48-1-12$,

or $13x=91$;

$\therefore$ (18. Cor. 2.) $x=\frac{91}{13}=7$.

13. Given $\frac{x-1}{7}+\frac{23-x}{5}=7-\frac{4+x}{4}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $4\times5\times7=140$,

$$20x-20+644-28x=980-140-35x;$$

$\therefore$ (17) by trans^n, $20x-28x+35x=980-140+20-644$,

or $27x=216$

$\therefore$ (18. Cor. 2.) $x=\frac{216}{27}=8$.

14. Given $\frac{7x+5}{3}-\frac{16+4x}{5}+6=\frac{3x+9}{2}$, to find the value of x.

(18. Cor. 1.) multiplying both sides by $2\times3\times5=30$,

$$70x+50-96-24x+180=45x+135;$$

$\therefore$ (17) by trans^n, $70x-24x-45x=135+96-50-180$,

or $x=1$.

15. Given $\frac{3x+4}{5} - \frac{7x-3}{2} = \frac{x-16}{4}$, to find the value of x.

(18. Cor. 1.) multiplying by 20, the least common multiple of 2, 4, and 5, .

$$12x+16-70x+30=5x-80;$$

$\therefore$ (17) by transposition, $16+30+80=5x+70x-12x$,

$$\text{or } 126=63x;$$

$$\therefore \text{(18. Cor. 2.) } \frac{126}{63} = 2 = x.$$

16. Given $\frac{17-3x}{5} - \frac{4x+2}{3} = 5-6x+\frac{7x+14}{3}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $3 \times 5 = 15$,

$$51-9x-20x-10=75-90x+35x+70;$$

$\therefore$ (17) by trans^n, $90x-35x-20x-9x=75+70+10-51$,

$$\text{or } 26x=104;$$

$$\therefore \text{(18. Cor. 2.) } x=\frac{104}{26} = 4.$$

17. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $2 \times 5 \times 7 = 70$,

$$70x-42x+42+280=700-35x-60x+80+56x-56;$$

$\therefore$ (17) by transposition,

$$70x+35x+60x-42x-56x=700+80-56-280-42,$$

$$\text{or } 67x=402;$$

$$\therefore \text{(18. Cor. 2.) } x=\frac{402}{57} = 6.$$

18. Given $\frac{4x-21}{9}+3\frac{3}{4}+\frac{57-3x}{4}=241-\frac{5x-96}{12}$ $-11x$, to find the value of x.

(18. Cor. 1.) multiplying by 36, the least common multiple of 4, 9, and 12,

$16x-84+135+513-27x=8676-15x+288-396x$;

$\therefore$ (17) by transposition,

$16x+15x+396x-27x=8676+288+84-135-513$,

or $400x=8400$;

$\therefore$ (18. Cor. 2.) $x=\frac{8400}{400}=21$.

19. Given $\frac{6x+18}{13}-4\frac{5}{6}-\frac{11-3x}{36}=5x-48-\frac{13-x}{12}$ $-\frac{21-2x}{18}$, to find the value of x.

(18. Cor. 1.) multiplying by 36×13, the least common multiple of the denominators,

$216x+648-2262-143+39x=2340x-22464-507+39x-546+52x$;

$\therefore$ (17) by transposition, $648+22464+507+546-2262-143=2340x+39x+52x-216x-39x$,

or $21760=2176x$;

$\therefore$ (18. Cor. 2.) $\frac{21760}{2176}=10=x$.

20. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$, to find the value of x.

Multiplying both sides of the equation by 21,

$$7x+16-\frac{21x+168}{4x-11}=7x;$$

$\therefore$ (17. Cor. 3.) $16=\frac{21x+168}{4x-11}$;

$\therefore$ (18. Cor. 1.) $64x - 176 = 21x + 168$;

$\therefore$ (17) by transposition, $64x - 21x = 168 + 176$,

or $43x = 344$;

$\therefore$ (18. Cor. 2.) $x = \frac{344}{43} = 8$.

21. Given $\frac{6x+7}{9} + \frac{7x-13}{6x+3} = \frac{2x+4}{3}$, to find the value of x.

Multiplying both sides of the equation by 9,

$$6x + 7 + \frac{21x-39}{2x+1} = 6x + 12;$$

$\therefore$ (17. Cor. 3.) $\frac{21x-39}{2x+1} = 5$;

$\therefore$ (18. Cor. 1.) $21x - 39 = 10x + 5$;

$\therefore$ (17) by transposition, $21x - 10x = 39 + 5$,

or $11x = 44$;

$\therefore$ (18. Cor. 2.) $x = \frac{44}{11} = 4$.

22. Given $12 - x : \frac{x}{2} :: 4 : 1$, to find the value of x.

(21) Since the product of the extremes is equal to the product of the means,

$$12 - x = 4 \times \frac{x}{2} = 2x;$$

$\therefore$ (17) by transposition, $12 = 2x + x = 3x$,

and (18. Cor. 2.) $\frac{12}{3} = 4 = x$.

23. Given $\frac{5x+4}{2} : \frac{18-x}{4} :: 7 : 4$, to find the value of x.

$$(21)\ \frac{5x+4}{2}\times 4=\frac{18-x}{4}\times 7,$$

$$\text{or } 10x+8=\frac{126-7x}{4};$$

$$\therefore \text{ (18. Cor. 2.) } 40x+32=126-7x;$$

$$\therefore \text{ (17) by transposition, } 40x+7x=126-32,$$

$$\text{or } 47x=94;$$

$$\therefore x=\frac{94}{47}=2.$$

24. Given $\sqrt{4x+16}=12$, to find the value of x.

$$\text{(19) squaring both sides of the equation, } 4x+16=144;$$

$$\therefore \text{ (17) by transposition, } 4x=144-16=128;$$

$$\therefore \text{ (18. Cor. 2.) } x=\frac{128}{4}=32.$$

25. Given $\sqrt[3]{2x+3}+4=7$, to find the value of x.

$$\text{(17) by transposition, } \sqrt[3]{2x+3}=7-4=3;$$

$$\therefore \text{ (19) cubing both sides of the equation, } 2x+3=27;$$

$$\therefore \text{ (17) by transposition, } 2x=27-3=24,$$

$$\text{and (18. Cor. 2.) } x=\frac{24}{2}=12.$$

26. Given $\sqrt{12+x}=2+\sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$12+x=4+4\sqrt{x}+x;$$

$$\therefore \text{ (17. Cor. 3.) } 8=4\sqrt{x},$$

$$\text{and (18. Cor. 2.) } 2=\sqrt{x};$$

$$\therefore \text{ (19) } 4=x.$$

27. Given $\sqrt{x+40}=10-\sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$x+40=100-20\sqrt{x}+x;$$

$\therefore$ (17. Cor. 3.) $20\sqrt{x}=100-40=60$,

and (18. Cor. 2.) $\sqrt{x}=3$;

$\therefore$ (19) $x=9$.

28. Given $\sqrt{x-16}=8-\sqrt{x}$, to find the value of x.

(19) squaring both sides of the equation,

$$x-16=64-16\sqrt{x}+x;$$

$\therefore$ (17. Cor. 3.) $16\sqrt{x}=64+16=80$;

$\therefore$ (18. Cor. 2.) $\sqrt{x}=5$,

and (19) $x=25$.

29. Given $\sqrt{x-24}=\sqrt{x}-2$, to find the value of x.

(19) squaring both sides of the equation,

$$x-24=x-4\sqrt{x}+4;$$

$\therefore$ (17. Cor. 3.) $4\sqrt{x}=24+4=28$;

$\therefore$ (18. Cor. 2.) $\sqrt{x}=7$,

and (19) $x=49$.

30. Given $\sqrt{5}\times\sqrt{x+2}=\sqrt{5x}+2$.

(19) squaring both sides of the equation,

$$5x+10=5x+4\sqrt{5x}+4;$$

$\therefore$ (17. Cor. 3.) $10-4=6=4\sqrt{5x}$;

$\therefore$ (18. Cor. 2.) $\frac{6}{4}=\frac{3}{2}=\sqrt{5x}$,

and (19) $\frac{9}{4}=5x$;

$\therefore$ (18. Cor. 2.) $\frac{9}{20}=x$.

31. Given $\frac{x-ax}{\sqrt{x}}=\frac{\sqrt{x}}{x}$, to find the value of x.

Multiplying both sides of the equation by $\sqrt{x}$,

$$x-ax=\frac{x}{x}=1,$$

$$\overline{1-a}\,.\,x=1,$$

and (18. Cor. 2.) $x=\frac{1}{1-a}$.

32. Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$, to find the value of x.

(18. Cor. 1.) multiplying both sides of the equation by $(\sqrt{x}+4)\times(\sqrt{x}+6)$,

$$x+34\sqrt{x}+168=x+42\sqrt{x}+152\,;$$

$\therefore$ (17. Cor. 3.) taking $(x+34\sqrt{x}+152)$ from each side of the equation,

$$16=8\sqrt{x}\,;$$

$\therefore$ (18. Cor. 2.) $2=\sqrt{x}$;

$\therefore$ (19) $4=x$.

33. Given $\frac{3x-1}{\sqrt{3x}+1}=1+\frac{\sqrt{3x}-1}{2}$, to find the value of x.

Since $3x-1=(\sqrt{3x}+1)\times(\sqrt{3x}-1)$;

$$\therefore\ \frac{3x-1}{\sqrt{3x}+1}=\sqrt{3x}-1\,;$$

$$\therefore\ \sqrt{3x}-1=1+\frac{\sqrt{3x}-1}{2},$$

and (17. Cor. 3.) taking $\frac{\sqrt{3x}-1}{2}$ from each side,

$$\frac{\sqrt{3x}-1}{2} = 1;$$

$\therefore$ (18. Cor. 1.) $\sqrt{3x}-1=2$;

$\therefore$ (17) by transposition, $\sqrt{3x}=2+1=3$;

$\therefore$ (19) squaring both sides, $3x=9$,

(18. Cor. 2.) $x=\frac{9}{3}=3$.

34. Given $x=\sqrt{a^2+x\sqrt{b^2+x^2}}-a$, to find the value of x.

(17) by transposition, $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}$;

$\therefore$ (19) squaring both sides, $x^2+2ax+a^2=a^2+x\sqrt{b^2+x^2}$;

$\therefore$ (17. Cor. 3.) $x^2+2ax=x\sqrt{b^2+x^2}$;

$\therefore$ (18. Cor. 2.) $x+2a=\sqrt{b^2+x^2}$,

and (19) squaring both sides, $x^2+4ax+4a^2=b^2+x^2$;

$\therefore$ (17. Cor. 3.) $4ax+4a^2=b^2$;

$\therefore$ (17) by transposition, $4ax=b^2-4a^2$;

$\therefore$ (18. Cor. 2.) $x=\frac{b^2-4a^2}{4a}$.

35. Given $\sqrt{2+x}+\sqrt{x}=\frac{4}{\sqrt{2+x}}$, to find the value of x.

(18. Cor. 1.) $2+x+\sqrt{2x+x^2}=4$;

$\therefore$ (17) by transposition, $\sqrt{2x+x^2}=4-2-x=2-x$,

and (19) squaring both sides, $2x+x^2=4-4x+x^2$;

$\therefore$ (17. Cor. 3.) $6x=4$;

$\therefore$ (18. Cor. 2.) $x=\frac{4}{6}=\frac{2}{3}$.

36. Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$, to find the value of x.

(18. Cor. 1.) $5+x+\sqrt{5x+x^2}=15$;

∴ (17) by transposition, $\sqrt{5x+x^2}=15-5-x=10-x$,

and (19) squaring both sides, $5x+x^2=100-20x+x^2$;

∴ (17. Cor. 3.) $25x=100$;

∴ (18. Cor. 2.) $x=\frac{100}{25}=4$.

37. Given $x+\sqrt{a^2+x^2}=\frac{2a^2}{\sqrt{a^2+x^2}}$, to find the value of x.

(18. Cor. 1.) $x\sqrt{a^2+x^2}+a^2+x^2=2a^2$;

∴ (17) by transposition, $x\sqrt{a^2+x^2}=2a^2-a^2-x^2=a^2-x^2$,

and (19) squaring both sides, $x^2 \cdot \overline{a^2+x^2}=a^4-2a^2x^2+x^4$,

or $a^2x^2+x^4=a^4-2a^2x^2+x^4$;

∴ (17. Cor. 3.) $3a^2x^2=a^4$,

and (18. Cor. 2.) $x^2=\frac{a^4}{3a^2}=\frac{a^2}{3}$;

∴ (19) $x=\frac{a}{\sqrt{3}}$.

SECT. II.

On the Solution of Simple Equations which involve more than one unknown Quantity.

(23.) IF the equations involve several unknown quantities, and definite values of these are required, there must necessarily be as many independent equations as there are unknown quantities. In which case, the values will be found by exterminating all the unknown quantities except one; and this may be done by either of the three following methods:

1. By equalizing the coefficients of the same unknown quantity in the several equations.
2. By substitution.
3. By equating different values of the same unknown quantity.

1. Of exterminating an unknown quantity by the first method in equations where two unknown quantities are concerned.

If the coefficient of either unknown quantity in one equation be contained a certain number of times exactly in the coefficient of the same unknown quantity in the other, multiply the former equation by that number, then add it to, or subtract it from, the other equation, according as the signs are different or the same, and an equation arises, in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{l}4x+y=34\\ \text{and } 4y+x=16\end{array}\right\}$ Here the coefficient of x in the second equation is contained 4 times exactly in the first; multiplying therefore the second equation by 4, and subtracting the first from it,

$$\begin{array}{rl} & 4x+16y=64, \\ \text{and} & \underline{4x+y\quad=34;} \\ \therefore & \quad 15y\quad=30, \text{ and } y=2. \end{array}$$

Having thus obtained a value of one of the unknown quantities, the other may be determined by substituting in either equation the value of the quantity found, and thus reducing the equation to one which contains only the other unknown quantity. Thus, from the second of the preceding equations, $x=16-4y=16-8=8$.

The values of x and y might be found in a similar manner, by multiplying the first equation by 4, and subtracting the second from it.

But if neither of the coefficients be a measure of the coefficient of the same unknown quantity in the other equation, multiply the first equation by the coefficient of one of the unknown quantities in the second equation, and the second equation by the coefficient of the same unknown quantity in the first. If the signs of the unknown quantity be alike in both, subtract one equation from the other; if unlike, add them together, and an equation arises in which only one unknown quantity is found.

Thus, if $\left.\begin{array}{l}2x+3y=23\\ \text{and } 5x-2y=10\end{array}\right\}$ In this case neither of the coefficients is a measure of the coefficient of the same unknown quantity in the other equation; and therefore, multiplying the first equation by 2, and the second by 3,

$$4x+6y=46,$$
$$\text{and } 15x-6y=30;$$

$\therefore$ by addition, $19x=76$, and $x=4$;
whence, as before, $3y=23-2x=23-8=15$,
and $y=5$.

The values of x and y might also be obtained, by multiplying the first equation by 5, and the second by 2, and then subtracting the second from the first.

2. By substitution.

Find the value of one of the unknown quantities, in terms of the other and known quantities, in the more simple of the two equations; and substitute this value instead of the quantity itself in the other equation; thus an equation is obtained in which there is only one unknown quantity.

Thus in the first of the preceding examples; from the second equation, $x=16-4y$; substituting therefore this value of x in the first equation,

$$4.\overline{16-4y}+y=34,$$
$$\text{or } 64-16y+y=34;$$
$$\therefore \text{ by transposition, } (64-34=)\ 30=15y,$$
$$\text{and therefore } 2=y;$$
$$\text{whence, as before, } x=8.$$

Here a value of x might have been obtained from the second equation, and substituted for it in the first; whence an equation would have arisen, involving only y; the value of which being found, that of x also might be determined, as before, by substitution.

Or a value of y might be determined from either equation, and substituted in the other; from which would arise an equation involving only x, the value of

which might be found; and therefore the value of y also might be obtained by substitution.

Again, in the second example; from the first equation is obtained

$$2x = 23 - 3y; \text{ and therefore } x = \frac{23 - 3y}{2};$$

substituting therefore this value in the second equation,

$$5 \cdot \frac{23 - 3y}{2} - 2y = 10,$$

$$\text{or } 115 - 15y - 4y = 20;$$

$$\therefore \text{ by transposition, } 115 - 20 = 15y + 4y,$$

$$\text{or } 95 = 19y;$$

$$\therefore 5 = y,$$

$$\text{and } x = \frac{23 - 3y}{2} = \frac{23 - 15}{2} = \frac{8}{2} = 4.$$

Here also a value of x might be obtained from the second equation, and substituted in the first, which would give an equation involving only y; or a value of y might be obtained from either equation, which substituted in the other would give an equation involving only x; the value of which might therefore be found, and consequently that of y might also be determined.

3. By equating different values of the same unknown quantity.

From each equation find the value of the same unknown quantity in terms of the other and known quantities; then, by equating the values so found, an equation arises containing only one unknown quantity.

Thus in the first of the preceding examples; from the first equation, $y = 34 - 4x$,

and from the second equation,

$$4y = 16 - x; \text{ and therefore } y = \frac{16 - x}{4};$$

$$\therefore \frac{16-x}{4}=34-4x;$$

consequently, $16-x=136-16x$;

∴ by transposition, $16x-x=136-16$,

or $15x=120$;

$\therefore x=8$,

nd $y=34-4x=34-32=2$, as before.

case also, two values of x are deducible from the ıations, which would give an equation involving ; and the value of y being determined, that of t also be found.

.in, in the second of the preceding examples;

from the first equation, $x=\frac{23-3y}{2}$,

and from the second, $x=\frac{10+2y}{5}$;

$$\therefore \frac{10+2y}{5}=\frac{23-3y}{2},$$

and $20+4y=115-15y$;

by transposition, $4y+15y=115-20$,

or $19y=95$;

$\therefore y=5$, and $x=4$, as before.

ʒain two values of y might have been found, ⁊ould have given an equation involving only x; m the solution of this new equation, a value ıd therefore of y, might be found.

EXAMPLES.

Given $\left.\begin{array}{l} 5x+4y=58 \\ \text{and } 3x+7y=67 \end{array}\right\}$, to find the values of x

Multiplying the second equation by 5, and the first by 3,

$$15x+35y=335,$$
$$\text{and } 15x+12y=174;$$

$$\therefore \text{ by subtraction,} \quad 23y=161,$$
$$\text{and} \quad y=7;$$
$$\text{whence, } 5x=58-4y=58-28=30,$$
$$\text{and therefore } x=6.$$

If the second equation had been multiplied by 4, and subtracted from the first when multiplied by 7, an equation would have arisen, involving only x, the value of which might be determined, and thence, by substitution, the value of y.

Second Method.

From the second equation, $3x= 67-7y$;

$$\therefore x=\frac{67-7y}{3}.$$

Substituting this value of x in the first equation,

$$5 \,.\, \frac{67-7y}{3}+4y=58,$$
$$\text{and } 335-35y+12y=174;$$
$$\therefore \text{ by transposition, } 335-174=35y-12y,$$
$$\text{or } 161=23y;$$
$$\therefore 7=y;$$

whence, as before, the value of x may be found. In the same manner, a value of x might be found from the first equation, which substituted in the second, would give an equation involving only y. Or a value of y might be obtained from either equation, and substituted for it in the other; whence an equation would arise involving only x, the value of which might be found, and therefore that of y also determined.

Third Method.

From the first equation, $5x = 58 - 4y$;

$$\therefore\ x = \frac{58-4y}{5}.$$

From the second, $x = \dfrac{67-7y}{3}$;

$$\therefore\ \frac{58-4y}{5} = \frac{67-7y}{3},$$

and $174 - 12y = 335 - 35y$;

by transposition, $35y - 12y = 335 - 174$,

or $23y = 161$;

$$\therefore\ y = 7;$$

whence, as before, $x = 6$.

In this case, two values of y might be deduced from the two equations; and from equating these, there would arise an equation involving x only; whose value being found, that of y also might be determined by substitution.

2. Given $\left.\begin{array}{r} 11x + 3y = 100 \\ \text{and } 4x - 7y = 4 \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 7, and the second by 3,

$$\begin{aligned} 77x + 21y &= 700, \\ \text{and } 12x - 21y &= 12; \end{aligned}$$

$\therefore$ by addition, $89x = 712$,

and $x = 8$;

whence $3y = 100 - 11x = 100 - 88 = 12$;

$$\therefore\ y = 4.$$

3. Given $\left.\begin{array}{l}\frac{x}{2}+\frac{y}{3}=7\\ \text{and } \frac{x}{3}+\frac{y}{2}=8\end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) clearing the equations of fractions, by multiplying each by 6,

$$3x+2y=42,$$
$$\text{and } 2x+3y=48;$$

and as the coefficients in this case are not aliquot parts, multiplying the first by 3, and the second by 2;

$$\therefore\ 9x+6y=126,$$
$$\text{and } 4x+6y=96;$$

$$\therefore \text{ by subtraction, } 5x \quad =30,$$
$$\text{and } x=6;$$
$$\text{whence } 2y=42-3x=42-18=24,$$
$$\text{and } y=12.$$

4. Given $\left.\begin{array}{l}\frac{x}{7}+7y=99\\ \text{and } \frac{y}{7}+7x=51\end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) multiplying each equation by 7,

$$\therefore\ x+49y=693,$$
$$\text{and } 49x+y=357;$$

$$\therefore \text{ by addition, } 50x+50y=1050,$$
$$\text{and therefore } x+y=21;$$
$$\text{but since } x+49y=693,$$

subtracting the upper equation from the lower,

$$48y=672;$$
$$\therefore\ y=14,$$
$$\text{whence } x=21-y=21-14=7.$$

5. Given $\left.\begin{aligned}\frac{x+2}{3}+8y&=31\\ \text{and } \frac{y+5}{4}+10x&=192\end{aligned}\right\}$, to find the values of x and y.

Clearing the first equation of fractions,

$$x+2+24y=93;$$

$\therefore$ by transposition, $x+24y=91$.

Clearing the second equation of fractions,

$$y+5+40x=768;$$

$\therefore$ by transposition, $40x+y=763$.

Multiplying the first equation by 40, and subtracting the second from it,

$$\begin{aligned}40x+960y&=3640;\\ 40x+y&=763;\\ \hline \therefore\ 959y&=2877,\end{aligned}$$

and $y=3$;

$\therefore$ $x=91-24y=91-72=19$.

6. Given $\left.\begin{aligned}\frac{2x-y}{2}+14&=18\\ \text{and } \frac{2y+x}{3}+16&=19\end{aligned}\right\}$, to find the values of x and y.

By transposition, $\frac{2x-y}{2}=4$, from the first equation,

and therefore $2x-y=8$.

Also, $\frac{2y+x}{3}=3$, from the second equation,

and therefore $2y+x=9$,

which, multiplied by 2, gives $2x+4y=18$;

but $2x-y=8$;

$\therefore$ by subtraction, $5y=10$,

and therefore $y=2$,

whence $x=9-2y=9-4=5$.

7. Given $\left.\begin{aligned}\frac{2x+3y}{6}+\frac{x}{3}&=8\\ \text{and } \frac{7y-3x}{2}-y&=11\end{aligned}\right\}$, to find the values of x and y.

Clearing the first equation of fractions, $2x+3y+2x=48$,

or $4x+3y=48$;

and clearing the second of fractions, $7y-3x-2y=22$,

or $5y-3x=22$.

Multiplying this by 4, and the preceding one by 3,

$9y+12x=144$,

and $20y-12x=88$;

$\therefore$ by addition, $29y=232$,

and therefore $y=8$,

whence $4x=48-3y=48-24=24$,

and $x=6$.

8. Given $\left.\begin{aligned}3x+\frac{7y}{2}&=22\\ \text{and } 11y-\frac{2x}{5}&=20\end{aligned}\right\}$, to find the values of x and y.

Clearing the first equation of fractions, $6x+7y=44$;

but from the second, $55y-2x=100$.

Multiplying this last by 3, $165y-6x=300$;

but $7y+6x=44$;

$\therefore$ by addition, $172y=344$,

and $y=2$.

Now $6x=44-7y=44-14=30$;

$\therefore\ x=5.$

9. Given $\left.\begin{array}{l} x+1 : y :: 5 : 3 \\ \text{and } \frac{2x}{3}-\frac{5-y}{2}=\frac{41}{12}-\frac{2x-1}{4} \end{array}\right\}$, to find the values of x and y.

From the second equation, (18. Cor. 1.) multiplied by 12,

$$8x-30+6y=41-6x+3;$$

$\therefore$ by transposition, $14x+6y=74$,

and $7x+3y=37$.

But from the first equation, $5y=3x+3$,

or $5y-3x=3$.

Multiplying this equation by 7, $35y-21x=21$,

and the former by 3, $9y+21x=111$;

$\therefore$ by addition, $44y=132$;

$\therefore\ y=3$;

$\therefore\ x+1=\frac{5y}{3}=5$, and therefore $x=4$.

10. Given $\left.\begin{array}{l} \frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4} \\ \text{and } \frac{2y+4}{3}-\frac{2x+y}{8}=\frac{x+13}{4} \end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 60,

$$12x-24-200+20x=15y-150;$$

and by transposition, $32x-15y=74$.

Also (18. Cor. 1.) multiplying the second equation by 24,

$$16y+32-6x-3y=6x+78;$$

$\therefore$ by transposition, $13y - 12x = 46$.

Now the coefficients of x have aliquot parts; multiplying therefore this by 8, and the preceding by 3,

$$104y - 96x = 368,$$

$$\text{and } 96x - 45y = 222;$$

$\therefore$ by addition, $59y = 590$; $\therefore y = 10$,

and $32x = 15y + 74 = 150 + 74 = 224$;

$$\therefore x = 7.$$

11. Given $\left.\begin{array}{l} 2y - \dfrac{x+3}{4} = 7 + \dfrac{3x-2y}{5} \\ \text{and } 4x - \dfrac{8-y}{3} = 24\frac{1}{2} - \dfrac{2x+1}{2} \end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) from the first equation,

$$40y - 5x - 15 = 140 + 12x - 8y;$$

$\therefore$ by transposition, $48y - 17x = 155$,

and from the second equation,

$$24x - 16 + 2y = 147 - 6x - 3;$$

$\therefore$ by transposition, $30x + 2y = 160$.

Multiplying this by 24, $48y + 720x = 3840$;

but $48y - 17x = 155$;

$\therefore$ by subtraction, $737x = 3685$,

and $x = 5$,

and $2y = 160 - 30x = 160 - 150 = 10$;

$$\therefore y = 5.$$

12. Given $\left.\begin{array}{l} \dfrac{2y}{18} - \dfrac{8x-2}{36} = 1 - \dfrac{4+y}{3} + \dfrac{x-y}{6} \\ \text{and } x : 3y :: 4 : 7 \end{array}\right\}$, to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{y}{9} - \frac{4x-1}{18} = 1 - \frac{4+y}{3} + \frac{x-y}{6};$$

and therefore (18. Cor. 1.) multiplying by 18,

$$2y - 4x + 1 = 18 - 24 - 6y + 3x - 3y;$$

$$\therefore \text{ by transposition, } 7 = 7x - 11y.$$

But from the second equation, $7x = 12y$.

Substituting therefore this value in the preceding equation,

$$7 = 12y - 11y = y,$$

$$\text{and therefore } x = \frac{12y}{7} = 12.$$

13. Given $x - \dfrac{3y-2+x}{11} = 1 + \dfrac{15x + \dfrac{4y}{3}}{33}$, and $\dfrac{3x+2y}{6} - \dfrac{y-5}{4} = \dfrac{11x+152}{12} - \dfrac{3y+1}{2}$, to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 33,

$$33x - 9y + 6 - 3x = 33 + 15x + \frac{4y}{3};$$

$$\text{by transposition, } 15x - 9y = 27 + \frac{4y}{3};$$

$$\therefore 45x - 27y = 81 + 4y,$$

$$\text{and } 45x - 31y = 81.$$

(18. Cor. 1.) multiplying the second equation by 12,

$$6x + 4y - 3y + 15 = 11x + 152 - 18y - 6;$$

$$\therefore \text{ by transposition, } 19y - 5x = 131.$$

Multiplying this by 9, $171y - 45x = 1179$;

but $45x - 31y = 81$;

$\therefore$ by addition, $140y = 1260$;

$\therefore\ y=9,$

and $5x=19y-131=171-131=40;$

$\therefore\ x=8.$

14. Given $\left.\begin{array}{l}\dfrac{80+3x}{15}=18\frac{1}{3}-\dfrac{4x+3y-8}{7} \\ \text{and } 10y+\dfrac{6x-35}{5}=55+10x\end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 105, the least common multiple of 3, 7, and 15,

$560+21x=1925-60x-45y+120;$

$\therefore$ by transposition, $81x+45y=1485;$

and dividing by 9, $9x+5y=165.$

Now from second equation, $50y+6x-35=275+50x;$

$\therefore$ by transposition, $50y-44x=310;$

and dividing by 2, $25y-22x=155;$

but multiplying the equation found above, by 5, $25y+45x=825;$

$\therefore$ by subtraction, $67x=670,$

and $x=10.$

Now $5y=165-9x=165-90=75;$

$\therefore\ y=15.$

15. Given $\left.\begin{array}{l} y+\dfrac{5x+2y}{6}-\dfrac{3y-12+8x}{5}=4-\dfrac{15+2x-4y}{3} \\ \text{and } \dfrac{7x+13-5y}{4}+x=2y-\dfrac{3x+2y-16}{3}\end{array}\right\}$ to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 30, the least common multiple of 3, 5, and 6,

$30y+25x+10y-18y+72-48x=120-150-20x+40y;$

whence, by transposition, $102 = 18y + 3x$;
and dividing by 3, $34 = 6y + x$.

Clearing the second equation of fractions,

$$21x + 39 - 15y + 12x = 24y - 12x - 8y + 64;$$

and by transposition, $45x - 31y = 25$.

Multiplying the former by 45, $45x + 270y = 1530$;

$\therefore$ by subtraction, $301y = 1505$,
and $y = 5$;
whence $x = 34 - 6y = 34 - 30 = 4$.

16. Given $1 + \frac{25 + 5y}{6} - \frac{7x - 6}{3} = 10 - \frac{3x - 10 + 7y}{12}$
and $\frac{12 - x}{9} : 5x - \frac{14 + y}{3} :: 1 : 8$,
to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by 12, the least common multiple of 3, 6, and 12,

$$12 + 50 + 10y - 28x + 24 = 120 - 3x + 10 - 7y;$$

$\therefore$ by transposition, $17y - 25x = 44$,

and (21) from the second equation,

$$\frac{96 - 8x}{9} = 5x - \frac{14 + y}{3};$$

$\therefore$ $96 - 8x = 45x - 42 - 3y$;
and by transposition, $138 = 53x - 3y$.

Multiplying this equation by 17, and the one found above by 3,

$$51y - 75x = 132,$$
and $$-51y + 901x = 2346;$$

$\therefore$ by addition, $826x = 2478$,
and $x = 3$.

Now $3y = 53x - 138 = 159 - 138 = 21$;
$\therefore$ $y = 7$.

17. Given $\left.\begin{array}{l}\frac{4x}{x^2}+\frac{5y}{y^2}=\frac{9}{y}-1 \\ \text{and } \frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2}\end{array}\right\}$, to find the values of x and y.

Reducing the first equation to lower terms,

$$\frac{4}{x}+\frac{5}{y}=\frac{9}{y}-1;$$

$$\therefore \text{ by transposition, } \frac{4}{x}-\frac{4}{y}=-1;$$

and from second equation, by trans^n, $-\frac{2}{x}+\frac{4}{y}=\frac{3}{2}$;

$$\therefore \text{ by addition, } \frac{2}{x}=\frac{1}{2};$$

$$\therefore 4=x,$$

$$\text{and } \frac{4}{y}=\frac{4}{x}+1=2;$$

$$\therefore 2y=4, \text{ and } y=2.$$

18. Given $\left.\begin{array}{l}x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2} \\ \text{and } y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}\end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by 2,

$$2x-\frac{4y-2x}{23-x}=40-59+2x;$$

$$\therefore \text{ by transposition, } 19=\frac{4y-2x}{23-x},$$

$$\text{and } 437-19x=4y-2x;$$

$$\text{and by transposition, } 437=17x+4y.$$

Also from second equation, $3y+\frac{3y-9}{x-18}=90-73+3y$,

and (17. Cor. 3.) $\frac{3y-9}{x-18}=17$;

$\therefore\ 3y-9=17x-306$;

by transposition, $297=17x-3y$;

but $437=17x+4y$;

$\therefore$ by subtraction, $140=7y$,

and $20=y$,

and $17x=297+3y=297+60=357$;

$\therefore\ x=21$.

19. Given $\left.\begin{array}{l} 8x-\frac{16+60x}{3y-1}=\frac{16xy-107}{5+2y} \\ \text{and } 2+6y+9x=\frac{27x^2-12y^2+38}{3x-2y+1} \end{array}\right\}$, to find the values of x and y.

Multiplying the first equation by $5+2y$,

$$40x+16xy-\frac{80+300x+32y+120xy}{3y-1}=16xy-107,$$

$\therefore$ by trans^n, $40x+107=\frac{80+300x+32y+120xy}{3y-1}$;

and multiplying by $3y-1$,

$$120xy-40x+321y-107=80+300x+32y+120xy.$$

$\therefore$ (17. Cor. 3.) $289y-340x=187$.

And from the second equation,

$$27x^2-12y^2+15x+2y+2=27x^2-12y^2+38,$$

$\therefore$ by transposition, $15x+2y=36$;

whence, the coefficients of x having aliquot parts, multiplying the first equation by 3, and the second by 68,

$$867y - 1020x = 561,$$

$$\text{and } 136y + 1020x = 2448;$$

$$\therefore \text{ by addition, } 1003y = 3009,$$

$$\text{and } y = 3,$$

$$\text{and } 15x = 36 - 2y = 36 - 6 = 30;$$

$$\therefore x = 2.$$

20. Given $\left.\begin{array}{l} 3 - \dfrac{7 + \frac{2x}{y}}{5} = 5 - \dfrac{5x + 9}{3y} \\ \text{and } y - \dfrac{4 + 15y}{6x - 2} = \dfrac{2xy - \frac{107}{8}}{2x + 5} \end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) multiplying the first equation by $15y$,

$$\therefore 45y - 21y - 6x = 75y - 25x - 45;$$

and by transposition, $51y - 19x = 45$.

Multiplying the second equation by $2x + 5$,

$$2xy + 5y - \frac{8x + 20 + 30xy + 75y}{6x - 2} = 2xy - \frac{107}{8};$$

$$\therefore \text{(17. Cor. 3.) } 5y + \frac{107}{8} = \frac{8x + 20 + 30xy + 75y}{6x - 2};$$

and multiplying by $6x - 2$,

$$30xy - 10y + \frac{321x - 107}{4} = 8x + 20 + 30xy + 75y;$$

$$\therefore \text{(17. Cor. 3.) } \frac{321x - 107}{4} = 8x + 85y + 20,$$

$$\text{and } 321x - 107 = 32x + 340y + 80;$$

and by transposition, $-187 = 340y - 289x$.

The coefficients of y in this case having aliquot parts; multiplying the first by 20, and the last by 3,

$$1020y - 380x = 900,$$
$$\text{and } 1020y - 867x = -561;$$

$\therefore$ by subtraction, $487x = 1461$,

and $x = 3$;

consequently, $51y = 45 + 19x = 45 + 57 = 102$;

$\therefore\ y = 2$.

(24.) If there be three unknown quantities, their values may be found from three independent equations.

For from two of the equations, a third, which involves only two of the unknown quantities, may be deduced by the preceding rules; and from the remaining equation, and one of the others, another which contains the same two unknown quantities. Having therefore two equations, which involve only two unknown quantities, these may be determined; and, by substituting their values in any of the original equations, that of the third quantity will be obtained. In some particular equations, two unknown quantities may be exterminated at once.

EXAMPLES.

1. Given $\left.\begin{aligned} x+y+z &= 31 \\ x+y-z &= 25 \\ x-y-z &= 9 \end{aligned}\right\}$, to find the values of x, y and z.

Adding the first and third equations, $2x = 40$;

$\therefore\ x = 20$.

Subtracting the second from the first, $2z=6$;

$\therefore\ z=3$;

and subtracting the third from the second, $2y=16$;

$\therefore\ y=8$.

2. Given $\left.\begin{aligned} x+y+z&=29 \\ x+2y+3z&=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}&=10 \end{aligned}\right\}$, to find the values of x, y and z.

Subtracting the first equation from the second,

$$y+2z=33.$$

(18. Cor. 1.) multiplying the third equation by 12, the least common multiple of 2, 3, and 4,

$$6x+4y+3z=120;$$

multiplying the first equation by 6, $6x+6y+6z=174$;

$\therefore$ by subtraction, $2y+3z=54$;

but $2y+4z=66$;

$\therefore$ by subtraction, $z=12$,

and $y=33-2z=33-24=9$;

also $x=29-y-z=29-9-12=8$.

In like manner, had the first equation been multiplied by 2, and subtracted from the second, an equation would have resulted, involving only x and z; and had it been multiplied by 4, and subtracted from the third when cleared of fractions, another equation would have been obtained, involving also x and z; whence, by the preceding rules, the values of x and z would be found, and consequently the value of y also, by substitution. Or if the first equation be multiplied by 3, and the second subtracted from it, an equation would arise involving only x and y; and if the first, when multiplied by 3, be subtracted from the third when cleared

of fractions, another would arise involving only x and y; whence the values of x and y might be determined. And hence the third, that of z, might be found.

Second Method.

From the first equation, $x=29-y-z$:

$\therefore$ substituting this value of x in the second equation,

$$29-y-z+2y+3z=62;$$

$\therefore$ by transposition, $y=33-2z$.

Also substituting, in the third equation, the value of x found from the first,

$$\frac{29-y-z}{2}+\frac{y}{3}+\frac{z}{4}=10;$$

$\therefore$ (18. Cor. 1.) $174-6y-6z+4y+3z=120$,

and by transposition, $54=3z+2y$;

in which, substituting the value of y found above,

$$54=3z+66-4z;$$

$\therefore$ by transposition, $z=12$;

whence $y=33-2z=33-24=9$,

and $x=29-y-z=29-9-12=8$.

It may be observed, that there will be the same variety of solution, as in the last case, according as x, y, or z, is exterminated.

Third Method.

From the first equation, $x=29-y-z$,

and from the second, $x=62-2y-3z$,

$\therefore$ $29-y-z=62-2y-3z$,

and by transposition, $y=33-2z$.

Again, from the third equation, $x=20-\frac{2y}{3}-\frac{z}{2}$;

$$\therefore\ 29-y-z=20-\frac{2y}{3}-\frac{z}{2};$$

and by transposition, $9 - \frac{z}{2} = \frac{y}{3}$;

$$\therefore\ 27 - \frac{3z}{2} = y;$$

whence $27 - \frac{3z}{2} = 33 - 2z$;

$\therefore$ by transposition, $\frac{z}{2} = 6$,

and $z = 12$;

whence $y = 9$, and $x = 8$, as before.

The same observation applies to this solution, as did to the last.

3. Given $\left.\begin{array}{l} \frac{4x+3y+z}{10} - \frac{2y+2z-x+1}{15} = 5 + \frac{x-z-5}{5} \\ \text{and } \frac{9x+5y-2z}{12} - \frac{2x+y-3z}{4} = \frac{7y+z+3}{11} + \frac{1}{6} \\ \text{and } \frac{5y+3z}{4} - \frac{2x+3y-z}{12} + 2z = y - 1 + \frac{3x+2y+7}{6} \end{array}\right\}$

to find the values of x, y, and z.

Multiplying the first equation by 30, the least common multiple of 5, 10, and 15,

$$12x + 9y + 3z - 4y - 4z + 2x - 2 = 150 + 6x - 6z - 30;$$

$\therefore$ by transposition, $8x + 5y + 5z = 122$.

Again, multiplying the second equation by 132, the least common multiple of 4, 6, 11, 12,

$$99x + 55y - 22z - 66x - 33y + 99z = 84y + 12z + 36 + 22;$$

$\therefore$ by transposition, $33x - 62y + 65z = 58$.

Again, multiplying the third equation by 12, the least common multiple of 12, 6, 4,

$$15y + 9z - 2x - 3y + z + 24z = 12y - 12 + 6x + 4y + 14;$$

$\therefore$ by transposition, $8x+4y-34z=-2$;
but from the first equation, $8x+5y+5z=122$;

$\therefore$ by subtraction, $y+39z=124$.

Also the third equation being divided by 2,

$$4x+2y-17z=-1.$$

Multiplying this by 33, and the second by 4,

$$132x+66y-561z=-33,$$
and $132x-248y+260z=232$;

$\therefore$ by subtraction, $314y-821z=-265$;
but $314y+12246z=38936$

(by multiplying the equation found above by 314);

$\therefore$ by subtraction, $13067z=39201$,

and therefore $z=3$;

whence $y=124-39z=124-117=7$,

and $4x=17z-2y-1=51-14-1=36$;

$$\therefore x=9.$$

(25.) If there be four unknown quantities, their values may be found from four independent equations. For from the four given equations, by the preceding rules, three may be deduced which involve only three unknown quantities, the values of which may be found by the last Article; and hence the fourth may be found, by substituting in any of the four given equations, the values of the three quantities determined.

If there be n unknown quantities and n independent equations, the values of those quantities may be found in a similar manner. For from the n given equations, $n-1$ may be deduced, involving only $n-1$ unknown quantities; and from these $n-1$, $n-2$ may be obtained, involving only $n-2$ unknown quantities; and

so on, till only one equation remains, involving one unknown quantity; which being found, the values of all the rest may be determined by substitution.

(26.) If there be more unknown quantities than independent equations, some of these quantities cannot be found except in terms of the others; and by assuming values of these others, we may obtain an infinite number of corresponding values of the former quantities, which will satisfy the conditions proposed.

But if there be fewer unknown quantities than independent equations, the values of the unknown quantities may be found from the different equations; and if these values be the same, some of the equations are unnecessary; if different, the equations are incongruous.

Sect. III.

On the Solution of Pure-Quadratics, and others which may be solved without completing the Square.

(27.) When the terms of an equation involve the square of the unknown quantity only, the value of the square will be found by the preceding articles; and extracting the root on each side of the equation, the unknown quantity itself will be determined.

In the same way any pure equation may be solved; for the power of the unknown quantity standing alone on one side of the equation, the known quantities being transposed to the other, the simple unknown quantity will be determined by extracting the root.

And by the same process, any equation containing the powers of a function of the unknown quantity, or containing the powers of two unknown quantities, may frequently be reduced to lower dimensions.

Examples.

1. Given $x^2 - 17 = 130 - 2x^2$, to find the values of x.

By transposition, $3x^2 = 147$;

$\therefore\quad x^2 = 49$,

and $x = \pm 7$.*

* The square root of a quantity may be either + or —, and consequently all quadratic equations admit of two solutions. Thus, $+7 \times +7$, and -7×-7, are both equal to 49; and both, when substituted for x in the original equation, answer the conditions required.

2. Given $\left.\begin{array}{l} x+y : x :: 5 : 3 \\ \text{and } xy=6 \end{array}\right\}$, to find the values of x and y.

$$\text{Since } x+y : x :: 5 : 3;$$

$$\therefore \text{ (\textit{Wood's Alg.} 180.) } y : x :: 2 : 3;$$

$$\therefore \text{ (21) } 3y=2x, \text{ and } y=\frac{2x}{3}.$$

Substitute this value in the second equation,

$$\therefore \frac{2x^2}{3}=6,$$

$$\text{and } x^2=9;$$

therefore, extracting the square root, $x=\pm 3$,

$$\text{whence } y=\frac{2x}{3}=\pm 2.$$

3. Given $\left.\begin{array}{l} x+y : x-y :: 3 : 1 \\ \text{and } x^3-y^3=56 \end{array}\right\}$, to find the values of x and y.

From the first equation, (*Alg.* 182.)

$$2x : 2y :: 4 : 2;$$

$$\therefore \text{ (\textit{Alg.} 184.) } x : y :: 2 : 1,$$

$$\text{and } x=2y.$$

Substituting this value of x in the second equation,

$$\therefore 8y^3-y^3=56,$$

$$\text{or } 7y^3=56;$$

$$\therefore y^3=8,$$

$$\text{and } y=2,$$

$$\text{whence } x=2y=4.$$

4. Given $\left.\begin{array}{l} x-y : x :: 5 : 6 \\ \text{and } xy^2=384 \end{array}\right\}$, to find the values of x and y.

(By *Alg.* 177 & 181.) $x : y :: 6 : 1$;

$\therefore$ (21) $x=6y$.

Substituting this value of x in the second equation,

$$\text{and } 6y^3=384;$$
$$\therefore y^3=64,$$
$$\text{whence } y=4,$$
$$\text{and } x=6y=24.$$

5. Given $\left.\begin{array}{l} x+y : x :: 7 : 5 \\ \text{and } xy+y^2=126 \end{array}\right\}$, to find the values of x and y.

(*Alg.* 180.) $y : x :: 2 : 5$;

$\therefore$ (21) $2x=5y$,

$$\text{and } x=\frac{5y}{2}.$$

Substituting this value for x in the second equation,

$$\frac{5y^2}{2}+y^2=126;$$
$$\therefore 5y^2+2y^2=252,$$
$$\text{or } 7y^2=252;$$
$$\therefore y^2=36,$$
$$\text{and } y=\pm 6;$$
$$\therefore x=\frac{5y}{2}=\pm 15.$$

6. Given $\left.\begin{array}{l} \overline{x+y}|^2 : \overline{x-y}|^2 :: 64 : 1 \\ \text{and } xy=63 \end{array}\right\}$, to find the values of x and y.

(*Alg.* 188.) $x+y : x-y :: 8 : 1$;

$\therefore$ (*Alg.* 182.) $2x : 2y :: 9 : 7$,

and (*Alg.* 184.) $x : y :: 9 : 7$;

$\therefore$ (21) $7x=9y$,

$$\text{and } x=\frac{9y}{7}.$$

Substituting this for x in the second equation,

$$\frac{9y^2}{7}=63;$$

$$\therefore\ y^2=49,$$

$$\text{and}\ \ y=\pm 7;$$

$$\therefore\ x=\frac{9y}{7}=\pm 9.$$

7. Given $\left.\begin{array}{l} x^2+xy=12 \\ \text{and } y^2+xy=24 \end{array}\right\}$, to find the values of x and y.

Adding the two equations together,

$$x^2+2xy+y^2=36;$$

$\therefore$ extracting the square root, $x+y=\pm 6$.

$$\text{Now}\ \ x^2+xy=x\,.\,\overline{x+y}=\pm 6x;$$

$$\therefore\ \pm 6x=12,$$

$$\text{and}\ \ x=\pm 2;$$

$$\text{and therefore}\ \ y=\pm 6\mp 2=\pm 4.$$

8. Given $\left.\begin{array}{l} x+y=s \\ \text{and } x^2-y^2=d^2 \end{array}\right\}$, to find the values of x and y.

$$\text{Since}\ \ x^2-y^2=\overline{x+y}\,.\,\overline{x-y}=s\,.\,\overline{x-y};$$

$$\therefore\ s\,.\,\overline{x-y}=d^2,$$

$$\text{and}\ \ x-y=\frac{d^2}{s};$$

$$\text{but}\ \ x+y=s;$$

$$\therefore\ \text{by addition,}\ \ 2x=s+\frac{d^2}{s}=\frac{s^2+d^2}{s},$$

$$\text{and}\ \ x=\frac{s^2+d^2}{2s}.$$

and by subtraction, $2y = s - \frac{d^2}{s} = \frac{s^2 - d^2}{s}$;

$$\therefore\ y = \frac{s^2 - d^2}{2s}.$$

9. Given $\left.\begin{array}{l}\sqrt[3]{x} + \sqrt[3]{y} = 5 \\ \text{and } \sqrt[3]{x} - \sqrt[3]{y} = 1\end{array}\right\}$, to find the values of x and y.

$$\begin{aligned} \text{Adding the two equations, } 2\sqrt[3]{x} &= 6; \\ \therefore\ \sqrt[3]{x} &= 3, \\ \text{and } x &= 27, \\ \text{and subtracting the equations, } 2\sqrt[3]{y} &= 4; \\ \therefore\ \sqrt[3]{y} &= 2, \\ \text{and } y &= 8. \end{aligned}$$

10. Given $\left.\begin{array}{l} x^2 + y^2 = \frac{13}{x-y} \\ \text{and } \quad xy = \frac{6}{x-y}\end{array}\right\}$, to find the values of x and y.

From the first equation subtracting twice the second,

$$(x^2 - 2xy + y^2 =)\ \overline{x-y}|^2 = \frac{1}{x-y};$$

$$\therefore\ \overline{x-y}|^3 = 1, \text{ and } x - y = 1;$$

$$\therefore\ x^2 + y^2 = 13,$$

$$\text{and } \quad 2xy = 12;$$

$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 25,$$

$$\text{and } \quad x + y = \pm 5;$$

$$\text{but } \quad x - y = \quad 1;$$

$$\therefore \text{ by addition, } \quad 2x = 6, \text{ or } -4,$$

$$\text{and } x = 3, \text{ or } -2,$$

$$\text{and by subtraction, } 2y = 4, \text{ or } -6;$$

$$\therefore\ y = 2, \text{ or } -3.$$

11. Given $\left.\begin{array}{r}x^2 - xy = 48y\\ \text{and } xy - y^2 = 3x\end{array}\right\}$, to find the values of x and y.

Dividing the first equation by x, $x - y = \dfrac{48y}{x}$;

and the second by y, $x - y = \dfrac{3x}{y}$;

$$\therefore\ \frac{48y}{x} = \frac{3x}{y};$$

$$\therefore\ 48y^2 = 3x^2,$$

$$\text{and } 16y^2 = x^2;$$

$$\text{consequently, } \pm 4y = x;$$

$$\text{and first, suppose } + 4y = x;$$

$$\therefore\ (x - y =)\ 3y = \left(\frac{48y}{x} = \frac{48y}{4y} =\right) 12;\quad \therefore\ y = 4;$$

$$\therefore\ x = 4y = 16.$$

But if $x = -4y$,

$$(x - y =) - 5y = \left(\frac{48y}{x} = \frac{48y}{-4y} =\right) -12;$$

$$\therefore\ y = \frac{12}{5},$$

$$\text{and } x = -4y = -\frac{48}{5}.$$

12. Given $\left.\begin{array}{r}\dfrac{xy}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} = 48\\ \text{and } \dfrac{xy}{\sqrt{x}} = 24\end{array}\right\}$, to find the values of x and y.

Dividing the first equation by the second,

$$\left(\frac{\sqrt{x}}{x^{\frac{1}{2}}y^{-\frac{1}{2}}} =\right) y^{\frac{1}{2}} = 2;$$

$$\therefore\ y = 4;$$

whence, from the second equation, $\frac{4x}{\sqrt{x}}=4\sqrt{x}=24$;

$$\therefore \sqrt{x}=6,$$

$$\text{and } x=36.$$

13. Given $\dfrac{x^2+3x-7}{x+2+\dfrac{18}{x}}=1$, to find the values of x.

Clearing the equation of fractions,

$$x^2+3x-7=x+2+\frac{18}{x};$$

$$\therefore \text{ by transposition, } x^2+2x=9+\frac{18}{x},$$

$$\text{or } x\,.\,\overline{x+2}=9\,.\,\overline{1+\frac{2}{x}}=\frac{9}{x}\,.\,\overline{x+2};$$

$$\therefore x=\frac{9}{x},$$

$$\text{and } x^2=9;$$

$$\therefore x=\pm 3.$$

14. Given $\left.\begin{array}{l} x^{\frac{2}{3}}+y^{\frac{2}{3}}=13 \\ \text{and } x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ \ 5 \end{array}\right\}$, to find the values of x and y.

Squaring the second equation, $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=25$;

but $x^{\frac{2}{3}} \qquad +y^{\frac{2}{3}}=13$;

$\therefore$ by subtraction, $\qquad 2x^{\frac{1}{3}}y^{\frac{1}{3}}=12.$

Subtracting this from first equation, $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=1$;

$\therefore$ extracting the root, $x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm 1$;

but $x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ \ 5$;

$\therefore$ by addition, $\qquad 2x^{\frac{1}{3}}=6$, or 4,

$$\text{and } x^{\frac{1}{3}} = 3, \text{ or } 2;$$
$$\text{whence } x = 27, \text{ or } 8;$$
$$\text{but by subtraction, } 2y^{\frac{1}{3}} = 4, \text{ or } 6,$$
$$\text{and } y^{\frac{1}{3}} = 2, \text{ or } 3;$$
$$\therefore y = 8, \text{ or } 27.$$

15. Given $\left.\begin{array}{l} xy^2 + y = 21 \\ \text{and } x^2y^4 + y^2 = 333 \end{array}\right\}$, to find the values of x and y.

$$\text{Squaring the first equation, } x^2y^4 + 2xy^3 + y^2 = 441;$$
$$\text{but } x^2y^4 + y^2 = 333;$$
$$\therefore \text{ by subtraction, } 2xy^3 = 108;$$

Subtracting this from the second equation,

$$x^2y^4 - 2xy^3 + y^2 = 225;$$
$$\text{extracting the root, } xy^2 - y = \pm 15;$$
$$\text{but } xy^2 + y = 21;$$
$$\therefore \text{ by subtraction, } 2y = 6, \text{ or } 36,$$
$$\text{and } y = 3, \text{ or } 18;$$
$$\text{but by addition, } 2xy^2 = 36, \text{ or } 6;$$
$$\therefore xy^2 = 18, \text{ or } 3,$$
$$\text{and therefore } x = \frac{18}{9}, \text{ or } \frac{3}{324},$$
$$= 2, \text{ or } \frac{1}{108}.$$

16. Given $\left.\begin{array}{l} x^2y + xy^2 = 180 \\ \text{and } x^3 + y^3 = 189 \end{array}\right\}$, to find the values of x and y.

Adding 3 times the first equation to the second,

$$x^3 + 3x^2y + 3xy^2 + y^3 = 729;$$

whence, extracting the cube root, $x + y = 9$.

$$\text{Now } x^2y + xy^2 = \overline{x+y} \,.\, xy = 180;$$

$$\therefore \text{ by substitution, } 9xy = 180,$$

$$\text{and } xy = 20.$$

$$\text{But } x^2 + 2xy + y^2 = 81,$$

$$\text{and } \quad 4xy \quad = 80;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 1;$$

$$\therefore \text{ extracting the square root, } x - y = \pm 1,$$

$$\text{and } x + y = 9;$$

$$\therefore \text{ by addition, } \quad 2x = 10, \text{ or } 8;$$

$$\therefore \quad x = 5, \text{ or } 4;$$

$$\text{and by subtraction, } 2y = 8, \text{ or } 10;$$

$$\therefore y = 4, \text{ or } 5.$$

17. Given $\left.\begin{array}{l} x + \sqrt{xy} + y = 19 \\ \text{and } x^2 + xy + y^2 = 133 \end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first,

$$x - \sqrt{xy} + y = 7;$$

$$\text{but } x + \sqrt{xy} + y = 19;$$

$$\therefore \text{ by addition, } 2x + 2y = 26;$$

$$\text{and } \quad x + y = 13;$$

$$\text{and by subtraction, } 2\sqrt{xy} = 12;$$

$$\therefore \sqrt{xy} = 6,$$

$$\text{and } xy = 36.$$

$$\text{Now from the second equation, } x^2 + xy + y^2 = 133;$$

$$\text{and from the last, } \quad 3xy \quad = 108;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 25;$$

and extracting the square root, $x-y=\pm 5$;

but $x+y= 13$;

$\therefore$ by addition, $2x = 18$, or 8,

and $x = 9$, or 4;

but by subtraction, $2y= 8$, or 18,

and $y= 4$, or 9.

18. Given $\dfrac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}=b$, to find the values of x.

Since the value of a fraction is not altered if the numerator and denominator be multiplied by the same quantity, (*Wood's Alg.* 89.) Multiply therefore by $a-\sqrt{a^2-x^2}$,

$$\text{and } \frac{\overline{a-\sqrt{a^2-x^2}}|^2}{x^2}=b.$$

$$\text{Extracting the root, } \frac{a-\sqrt{a^2-x^2}}{x}=\pm\sqrt{b},$$

$$\text{and } a-\sqrt{a^2-x^2}=\pm\sqrt{b}\,.\,x;$$

$$\therefore \text{ by transposition, } a\mp\sqrt{b}\,.\,x=\sqrt{a^2-x^2};$$

$$\text{and squaring both sides, } a^2\mp 2a\sqrt{b}\,.\,x+bx^2=a^2-x^2;$$

$$\therefore \text{ by transposition, } (b+1)\,.\,x^2=\pm 2a\sqrt{b}\,.\,x;$$

$$\text{and therefore } x=\pm\frac{2a\sqrt{b}}{1+b}.$$

19. Given $\left.\begin{array}{l} 3x-\dfrac{3x}{y}=y^2-y \\ \text{and } y^2+x=4 \end{array}\right\}$, to find the values of x and y.

$$\text{Reducing the first equation, } \frac{3x\,.\,(y-1)}{y}=y\,.\,(y-1);$$

and therefore (18. Cor. 2.) $\frac{3x}{y}=y,$

and $3x=y^2$.

Substituting this value in the second equation, $4x=4$;

$\therefore\ x=1,$

and $y^2=3x=3$;

$\therefore\ y=\pm\sqrt{3}.$

20. Given $\left.\begin{aligned} x^2+y\sqrt{xy}&=9\\ \text{and}\ y^2+x\sqrt{xy}&=18\end{aligned}\right\}$, to find the values of x and y.

The first equation is $x^{\frac{1}{2}}\times(x^{\frac{3}{2}}+y^{\frac{3}{2}})=9,$

and the second, $y^{\frac{1}{2}}\times(x^{\frac{3}{2}}+y^{\frac{3}{2}})=18.$

Dividing the second by the first, $\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}=2$;

$\therefore\ y^{\frac{1}{2}}=2x^{\frac{1}{2}},$

and $y=4x.$

Substituting this value in the first equation,

$x^2+4x\sqrt{4x^2}=9,$

or $9x^2=9$;

$\therefore\ x=\pm 1$;

consequently, $y=4x=\pm 4.$

21. Given $\left.\begin{aligned} \frac{x^2+2y^2}{\sqrt{y}}&=x+2xy\\ \text{and}\ x^2-2y^2&=256-x\sqrt{y}\end{aligned}\right\}$, to find the values of x and y.

From the first equation, $x^2+2y^2=x\sqrt{y}+2xy^{\frac{3}{2}}$;

$\therefore$ by transposition, $x^2-x\sqrt{y}=2xy^{\frac{3}{2}}-2y^2,$

or $x.(x-\sqrt{y})=2y^{\frac{3}{2}}.(x-\sqrt{y})$;

$\therefore\ x=2y^{\frac{3}{2}},$

$$\text{and } x\sqrt{y} = 2y^2;$$

$\therefore$ adding these equals to the second equation,

$$x^2 = 256,$$

$$\text{and therefore } x = \pm 16;$$

$$\therefore 2y^{\frac{3}{2}} = \pm 16,$$

$$\text{and } y^{\frac{3}{2}} = \pm 8;$$

$$\therefore y^{\frac{1}{2}} = \pm 2,$$

$$\text{and } y = 4.$$

22. Given $\left.\begin{array}{l} x^2y + xy^2 = 30 \\ \text{and } \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \end{array}\right\}$, to find the values of x and y.

From the second equation, $x + y = \dfrac{5xy}{6}$;

but from the first, $xy = \dfrac{30}{x+y}$.

Substituting this value in the other equation,

$$x+y = \frac{5}{6} \times \frac{30}{x+y} = \frac{25}{x+y};$$

$$\therefore \overline{x+y}|^2 = 25;$$

and extracting the root, $x+y = \pm 5$.

$$\text{Let } x+y = +5; \text{ then } xy = +6;$$

$$\text{whence } x^2 + 2xy + y^2 = 25,$$

$$\text{and } 4xy = 24;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 1$,

$$\text{and } x - y = \pm 1;$$

$$\text{but } x + y = 5;$$

$\therefore$ by addition, $2x = 6$, or 4; and $x = 3$, or 2;

by subtraction, $2y = 4$, or 6; and $y = 2$, or 3.

But if $x+y=-5$, then $xy=-6$;

$$\text{whence } x^2+2xy+y^2 = 25,$$
$$\text{and } 4xy = -24;$$

$\therefore$ by subtraction, $x^2-2xy+y^2 = 49$;
and extracting the root, $x-y=\pm 7$;
but $x+y=-5$;

$\therefore$ by addition, $2x=2$, or -12; and $x=1$, or -6;
by subtraction, $2y=-12$, or 2; and $y=-6$, or 1.

23. Given $\left.\begin{array}{l} x^2y+xy^2=6 \\ \text{and } x^3y^2+x^2y^3=12 \end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first, $xy=2$.

$$\text{But } x^2y+xy^2=\overline{x+y}\,.\,xy=6;$$
$$\therefore 2\,.\,\overline{x+y}=6,$$
$$\text{and } x+y=3;$$
$$\text{whence } x^2+2xy+y^2=9;$$
$$\text{but } 4xy = 8;$$

$\therefore$ by subtraction, $x^2-2xy+y^2=1$;
and extracting the root, $x-y=\pm 1$;
also $x+y=3$;

$\therefore$ by addition, $2x=4$, or 2;
$\therefore$ $x=2$, or 1;
and by subtraction, $2y=2$, or 4;
$\therefore$ $y=1$, or 2.

24. Given $\left.\begin{array}{l} x^3-y^3 : \overline{x-y}|^3 :: 61 : 1 \\ \text{and } xy=320 \end{array}\right\}$, to find the values of x and y.

Since $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: 61 : 1$;

$\therefore$ (*Alg.* 180.) $3x^2y-3xy^2 : \overline{x-y}|^3 :: 60 : 1$,

or $3xy\times\overline{x-y} : \overline{x-y}|^3 :: 60 : 1$;

$\therefore$ (*Alg.* 184.) $960 : \overline{x-y}|^2 :: 60 : 1$, dividing the first and second terms by $x-y$;

also $16 : \overline{x-y}|^2 :: 1 : 1$, dividing the first and third terms by 60;

$$\therefore \overline{x-y}|^2 = 16,$$

$$\text{and } x-y=\pm 4.$$

$$\begin{aligned} \text{But since } x^2-2xy+y^2 &= 16, \\ \text{and } 4xy &= 1280; \end{aligned}$$

$$\begin{aligned} \therefore \text{ by addition, } x^2+2xy+y^2 &= 1296; \\ \therefore \text{ extracting the root, } x+y &= \pm 36; \\ \text{but } x-y &= \pm 4; \end{aligned}$$

$$\begin{aligned} \therefore \text{ by addition, } 2x &= \pm 40, \text{ or } \pm 32, \\ \text{and } x &= \pm 20, \text{ or } \pm 16; \\ \text{but by subtraction, } 2y &= \pm 32, \text{ or } \pm 40, \\ \text{and } y &= \pm 16, \text{ or } \pm 20. \end{aligned}$$

25. Given $\left.\begin{aligned}(x^2+y^2)\times(x+y)&=2336 \\ \text{and } (x^2-y^2)\cdot(x-y)&=576\end{aligned}\right\}$, to find the values of x and y.

$$\begin{aligned} \text{From the first equation, } x^3+x^2y+xy^2+y^3 &= 2336; \\ \text{and from the second, } x^3-x^2y-xy^2+y^3 &= 576; \end{aligned}$$

$$\therefore \text{ by subtraction, } 2x^2y+2xy^2 = 1760;$$

adding this to the first equation,

$$x^3+3x^2y+3xy^2+y^3=4096;$$

$\therefore$ extracting the cube root, $x+y=16$,

$$\text{and } 2xy.\overline{x+y}=1760,$$
$$\text{or } 16\times 2xy=1760;$$
$$\therefore\ xy=55.$$

$$\text{Now } x^2+2xy+y^2=256,$$
$$\text{and } 4xy=220;$$

$$\therefore \text{ by subtraction, } x^2-2xy+y^2=36,$$
$$\text{and therefore } x-y=\pm 6;$$
$$\text{but } x+y=16;$$

$$\therefore \text{ by addition, } 2x=22, \text{ or } 10,$$
$$\text{and } x=11, \text{ or } 5;$$
$$\text{but by subtraction, } 2y=10, \text{ or } 22;$$
$$\therefore\ y=5, \text{ or } 11.$$

26. Given $\left.\begin{array}{l} x^3+y^3=\overline{x+y}.xy \\ \text{and } x^2y+xy^2=4xy \end{array}\right\}$, to find the values of x and y.

Dividing the second equation by xy, $x+y=4$;

$$\therefore\ x^3+3x^2y+3xy^2+y^3=64.$$

But from the first equation, $x^3-x^2y-xy^2+y^3=0$;

$$\therefore \text{ by subtraction, } 4x^2y+4xy^2=64;$$
$$\therefore\ \overline{x+y}.xy=16,$$
$$\text{and } xy=4.$$

$$\text{But } x^2+2xy+y^2=16,$$
$$\text{and } 4xy=16;$$

$$\therefore \text{ by subtraction, } x^2-2xy+y^2=0;$$
$$\text{and extracting the root, } x-y=0;$$
$$\text{but } x+y=4;$$

$$\therefore \text{ by addition, } 2x=4;$$

$$\text{and} \quad x = 2;$$
$$\text{but by subtraction,} \quad 2y = 4;$$
$$\therefore \quad y = 2.$$

27. Given $\left.\begin{array}{l}(x^2+y^2)\times(x+y) = \dfrac{15xy}{2} \\ \text{and } (x^4-y^4).(x^2+y^2) = \dfrac{75x^2y^2}{4}\end{array}\right\}$, to find the values of x and y.

Dividing the second by the first,

$$(x^2+y^2).(x-y) = \frac{5xy}{2}.$$

Again, dividing the first equation by this last,

$$\frac{x+y}{x-y} = 3;$$
$$\therefore \ x+y = 3x-3y;$$
$$\therefore \text{ by transposition,} \quad 4y = 2x,$$
$$\text{and } 2y = x;$$
$$\text{whence } (x^2+y^2).(x-y) = 5y^2 \times y = \frac{5y \times 2y}{2},$$
$$\text{or } y = 1.$$
$$\text{and therefore } x = 2y = 2.$$

28. Given $\left.\begin{array}{l}(x^2-y^2)\times(x-y) = 3xy \\ \text{and } (x^4-y^4)\times(x^2-y^2) = 45x^2y^2\end{array}\right\}$, to find the values of x and y.

Dividing the second equation by the first,

$$(x^2+y^2)\times(x+y) = 15xy,$$
$$\text{or } x^3+x^2y+xy^2+y^3 = 15xy;$$
$$\text{but from the first, } x^3-x^2y-xy^2+y^3 = 3xy;$$

$$\therefore \text{ by addition, } 2x^3 + 2y^3 = 18xy,$$
$$\text{and } x^3+y^3 = 9xy;$$

but by subtraction, $2x^2y+2xy^2=12xy$;

$\therefore$ dividing by $2xy$, $x+y=6$;

whence $x^3+3x^2y+3xy^2+y^3=216$;

but $x^3 \qquad\qquad +y^3=9xy$;

$\therefore$ by subtraction, $3x^2y+3xy^2 \quad =216-9xy$,

or $3.\overline{x+y}.xy=18xy=216-9xy$;

$\because$ $27xy=216$,

and $xy=8$.

Now $x^2+2xy+y^2=36$,

and $\quad 4xy \quad =32$;

$\therefore$ by subtraction, $x^2-2xy+y^2=4$;

and extracting the root, $x-y=\pm 2$;

but $x+y=6$;

$\therefore$ by addition, $2x=8$, or 4;

$\therefore$ $x=4$, or 2;

and by subtraction, $2y=4$, or 8;

$\therefore$ $y=2$, or 4.

Sect. IV.

Solution of Adfected Quadratics, involving only one unknown Quantity.

(28.) Let the terms be arranged on one side of the equation, according to the dimensions of the unknown quantity, beginning with the highest; and (17) the known quantities be transposed to the other side; then, if the square of the unknown quantity has any coefficient, either positive or negative, let all the terms be divided by this coefficient (13). If the square of half the coefficient of the second term be now added (11) to both sides of the equation*, that side which involves the unknown quantity will become a complete square; and (19) extracting the square root on both sides of the equation, a simple equation will be obtained, from which the values of the unknown quantity may be determined.

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* This is called completing the square; and that a complete square is thus obtained may be easily proved.

Let $x^2 \pm 2ax$ be the proposed quantity on one side, when the terms are arranged according to the form prescribed above; and suppose $y^2 =$ the quantity requisite to complete the square. Now the square of $x \pm d = x^2 \pm 2dx + d^2$, where it is evident that four times the product of the extreme terms is equal to the square of the middle term; and therefore, in order that $x^2 \pm 2ax + y^2$ may be a square, $4x^2y^2$ must be equal to $4a^2x^2$; therefore $y^2 = a^2 =$ the square of half the coefficient of the middle term.

It may be observed, that all equations may be solved as quadratics, by completing the squares, in which there are two terms involving the unknown quantity or any function of it, and the index of one is double that of the other. Thus, $x^6+px^3=q$, $x^{2n}-px^n=q$, $x^{\frac{n}{2}}+x^{\frac{n}{4}}=a$, $x^{3n}+ax^{\frac{3n}{2}}=b$, $\overline{x^2+px+q}|^2+\overline{x^2+px+q}=r$, are of the same form as quadratics, and the value of the unknown quantity may be determined in the same manner. Many equations also, in which more than one unknown quantity are involved, may in a similar manner be reduced to lower dimensions by completing the square, as $\overline{x^3+y^3}|^2+p\,.\,\overline{x^3+y^3}=r$. Instances of this kind occur in the following

Examples.

1. Given $x^2+4x=140$, to find the values of x.

Completing the square, $x^2+4x+4=144$;
and extracting the root, $x+2=\pm 12$;
whence, by transposition, $x=10$, or -14.

2. Given $x^2-6x+8=80$, to find the values of x.

By transposition, $x^2-6x=72$;
and completing the square, $x^2-6x+9=81$;
$\therefore$ extracting the root, $x-3=\pm 9$;
whence $x=12$, or -6.

3. Given $x^2-8x+10=19$, to find the values of x.

By transposition, $x^2-8x=9$;
and completing the square, $x^2-8x+16=25$;
$\therefore$ extracting the root, $x-4=\pm 5$,
and $x=9$, or -1.

4. Given $x^2-x-40=170$, to find the values of x.

By transposition, $x^2 - x = 210$;

and completing the square, $x^2 - x + \frac{1}{4} = 210 + \frac{1}{4} = \frac{841}{4}$,

$\therefore$ extracting the root, $x - \frac{1}{2} = \pm \frac{29}{2}$,

and $x = 15$, or -14.

5. Given $3x^2 + 2x - 9 = 76$, to find the values of x.

By transposition and division, $x^2 + \frac{2}{3}x = \frac{85}{3}$;

and completing the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{85}{3} + \frac{1}{9} = \frac{256}{9}$;

$\therefore$ extracting the root, $x + \frac{1}{3} = \pm \frac{16}{3}$;

whence $x = 5$, or $-\frac{17}{3}$.

6. Given $3x^2 - 9x - 4 = 80$, to find the values of x.

By transposition and division, $x^2 - 3x = \frac{84}{3} = 28$;

$\therefore$ completing the square, $x^2 - 3x + \frac{9}{4} = 28 + \frac{9}{4} = \frac{121}{4}$;

and extracting the root, $x - \frac{3}{2} = \pm \frac{11}{2}$;

consequently, $x = 7$, or -4.

7. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x.

(18. Cor. 1.) $4x^2 - 36 + x = 46x$;

$\therefore$ by transposition, $4x^2 - 45x = 36$,

and (18. Cor. 1.) $x^2 - \frac{45}{4} \cdot x = 9$;

therefore, completing the square,

$$x^2 - \frac{45}{4} \cdot x + \overline{\frac{45}{8}}\Big|^2 = \frac{2025}{64} + 9 = \frac{2601}{64};$$

$\therefore$ extracting the root, $x - \frac{45}{8} = \pm\frac{51}{8}$,

and $x = 12$, or $-\frac{3}{4}$.

8. Given $4x - \frac{14-x}{x+1} = 14$, to find the values of x.

Clearing the equation of fractions,

$$4x^2 + 4x - 14 + x = 14x + 14;$$

and therefore, by transposition, $4x^2 - 9x = 28$,

and (18. Cor. 1.) $x^2 - \frac{9}{4}x = 7$;

$\therefore$ completing the square, $x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 = 7 + \frac{81}{64} = \frac{529}{64}$;

and extracting the root, $x - \frac{9}{8} = \pm\frac{23}{8}$;

whence $x = 4$, or $-\frac{7}{4}$.

9. Given $3x - \frac{1121 - 4x}{x} = 2$, to find the values of x.

(18. Cor. 1.) $3x^2 - 1121 + 4x = 2x$;

$\therefore$ by transposition, $3x^2 + 2x = 1121$,

and (18. Cor. 1.) $x^2 + \frac{2}{3}x = \frac{1121}{3}$;

$\therefore$ completing the square, $x^2 + \frac{2}{3}x + \frac{1}{9} = \frac{1121}{3} + \frac{1}{9} = \frac{3364}{9}$;

and extracting the root, $x + \frac{1}{3} = \pm\frac{58}{3}$;

$\therefore x = 19$, or $-\frac{59}{3}$.

10. Given $\frac{8-x}{2} - \frac{2x-11}{x-3} = \frac{x-2}{6}$, to find the values of x.

Multiplying by 6, $24-3x-\frac{12x-66}{x-3}=x-2$;

by transposition, $26-4x=\frac{12x-66}{x-3}$,

and (18. Cor. 2.) $13-2x=\frac{6x-33}{x-3}$;

$\therefore\ 19x-39-2x^2=6x-33$;

$\therefore$ changing signs, and transposing,

$$2x^2-13x=-6,$$

$$\text{and } x^2-\frac{13}{2}x=-3;$$

and completing the square,

$$x^2-\frac{13}{2}x+\overline{\frac{13}{4}}\Big|^2=\frac{169}{16}-3=\frac{121}{16};$$

$\therefore$ extracting the root, $x-\frac{13}{4}=\pm\frac{11}{4}$;

$$\therefore\ x=6, \text{ or } \frac{1}{2}.$$

11. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find the values of x.

(18. Cor. 1.) $10x^2-36x+6=4x^2-12x+3x^2-15x+18$;

$\therefore$ by transposition, $3x^2-9x=12$,

$$\text{and } x^2-3x=4;$$

$\therefore$ completing the square, $x^2-3x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$;

and extracting the root, $x-\frac{3}{2}=\pm\frac{5}{2}$,

$$\text{and } x=4, \text{ or } -1.$$

12. Given $\frac{16}{x}-\frac{100-9x}{4x^2}=3$, to find the values of x.

(18. Cor. 1.) $64x-100+9x=12x^2$;
whence, by transposition, $12x^2-73x=-100$,

$$\text{and } x^2-\frac{73}{12}x=-\frac{100}{12};$$

and completing the square,

$$x^2-\frac{73}{12}.x+\overline{\frac{73}{24}}\Big|^2=\frac{5329}{576}-\frac{100}{12}=\frac{529}{576};$$

$$\therefore \text{ extracting the root, } x-\frac{73}{24}=\pm\frac{23}{24},$$

$$\text{and } x=4, \text{ or } \frac{25}{12}.$$

13. Given $3x-\dfrac{169-3x}{x}=29$, to find the values of x.

$$\text{Here } 3x^2-169+3x=29x;$$

$$\therefore \text{ by transposition, } 3x^2-26x=169,$$

$$\text{and } x^2-\frac{26}{3}x=\frac{169}{3};$$

therefore, completing the square,

$$x^2-\frac{26}{3}x+\frac{169}{9}=\frac{169}{3}+\frac{169}{9}=\frac{4}{9}169;$$

$$\text{and extracting the root, } x-\frac{13}{3}=\pm\frac{2}{3}13;$$

$$\therefore x=13, \text{ or } -\frac{13}{3}.$$

14. Given $16-\dfrac{2x^2}{3}=\dfrac{4x}{5}+7\dfrac{3}{5}$, to find the values of x.

Multiplying every term by $\dfrac{3}{2}$, $24-x^2=\dfrac{6x}{5}+\dfrac{57}{5}$;
$\therefore$ (17. Cor. 1.) and by transposition,

$$x^2+\frac{6x}{5}=24-\frac{57}{5}=\frac{63}{5};$$

and completing the square,

$$x^2+\frac{6x}{5}+\frac{9}{25}=\frac{63}{5}+\frac{9}{25}=\frac{324}{25};$$

$\therefore$ extracting the root, $x+\frac{3}{5}=\pm\frac{18}{5}$,

and $x=3$, or $-\frac{21}{5}$.

15. Given $\frac{10}{x}-\frac{14-2x}{x^2}=\frac{22}{9}$, to find the values of x.

(18. Cor. 1.) $10x-14+2x=\frac{22x^2}{9}$;

$\therefore$ (17. Cor. 1.) $\frac{22x^2}{9}-12x=-14$,

and $x^2-\frac{54}{11}x=-\frac{63}{11}$;

$\therefore$ completing the square,

$$x^2-\frac{54}{11}x+\left|\frac{27}{11}\right|^2=\frac{729}{121}-\frac{63}{11}=\frac{36}{11};$$

and extracting the root, $x-\frac{27}{11}=\pm\frac{6}{11}$;

$\therefore x=3$, or $\frac{21}{11}$.

16. Given $\frac{3x-4}{x-4}+1=10-\frac{x-2}{2}$, to find the values of x.

Clearing the equation of fractions,

$$6x-8+2x-8=20x-80-x^2+6x-8;$$

$\therefore$ by transposition, $x^2-18x=-72$;

and completing the square, $x^2-18x+81=81-72=9$;

whence, extracting the root, $x-9=\pm3$;

and therefore $x=12$, or 6.

17. Given $\frac{x+4}{3} - \frac{7-x}{x-3} = \frac{4x+7}{9} - 1$, to find the values of x.

Multiplying by 9, $3x+12-\frac{63-9x}{x-3}=4x+7-9$;

$\therefore$ by transposition, $14-x=\frac{63-9x}{x-3}$,

and $17x-x^2-42=63-9x$;

$\therefore$ by transposition, and changing signs, $x^2-26x=-105$;

completing the square, $x^2-26x+169=169-105=64$;

$\therefore$ extracting the root, $x-13=\pm 8$;

$\therefore x=21$, or 5.

18. Given $3x-\frac{3x-10}{9-2x}=2+\frac{6x^2-40}{2x-1}$, to find the values of x.

Multiplying by $2x-1$,

$$6x^2-3x-\frac{6x^2-23x+10}{9-2x}=4x-2+6x^2-40,$$

or $7x+\frac{6x^2-23x+10}{9-2x}=42$;

$\therefore 63x-14x^2+6x^2-23x+10=378-84x$;

by transposition, $124x-8x^2=368$,

and $x^2-\frac{31}{2}x=-46$;

$\therefore$ completing the square,

$$x^2-\frac{31}{2}x+\frac{961}{16}=\frac{961}{16}-46=\frac{225}{16};$$

$\therefore$ extracting the root, $x-\frac{31}{4}=\pm\frac{15}{4}$;

and therefore $x=\frac{23}{2}$, or 4.

19. Given $\frac{x}{5+x}+\frac{7}{6-4x}=\frac{11x}{11x-8}$, to find the values of x.

Multiplying by $\overline{5+x}$, $x+\frac{35+7x}{6-4x}=\frac{55x+11x^2}{11x-8}$;
and multiplying by $11x-8$,

$$11x^2-8x+\frac{329x+77x^2-280}{6-4x}=55x+11x^2,$$

$$\text{or } \frac{329x+77x^2-280}{6-4x}=63x;$$

$$\text{or, dividing by 7, } \frac{47x+11x^2-40}{6-4x}=9x;$$

$$\therefore\ 47x+11x^2-40=54x-36x^2;$$

$$\text{by transposition, } 47x^2-7x=40;$$

$$\therefore\ x^2-\frac{7}{47}x=\frac{40}{47};$$

and completing the square,

$$x^2-\frac{7}{47}+\overline{\frac{7}{94}}\Big|^2=\frac{49}{8336}+\frac{40}{47}=\frac{7569}{8336};$$

$$\text{and extracting the root, } x-\frac{7}{94}=\pm\frac{87}{94};$$

$$\therefore\ x=1,\ \text{or}\ -\frac{40}{47}.$$

20. Given $\frac{90}{x}-\frac{27}{x+2}=\frac{90}{x+1}$, to find the values of x.

Dividing every term by 9, $\frac{10}{x}-\frac{3}{x+2}=\frac{10}{x+1}$;
$\therefore$ (18. Cor. 1.) $10x^2+30x+20-3x^2-3x=10x^2+20x$;
$\therefore$ by transposition and (17. Cor. 1.) $3x^2-7x=20$,

$$\text{and } x^2-\frac{7}{3}x=\frac{20}{3};$$

completing the square, $x^2 - \frac{7}{3}x + \frac{49}{36} = \frac{20}{3} + \frac{49}{36} = \frac{289}{36}$;

and extracting the root, $x - \frac{7}{6} = \pm\frac{17}{6}$;

$\therefore\ x = 4$, or $-\frac{5}{3}$.

21. Given $\frac{1}{x^2 - 3x} + \frac{1}{x^2 + 4x} = \frac{9}{8x}$, to find the values of x.

Multiplying every term by x, $\frac{1}{x-3} + \frac{1}{x+4} = \frac{9}{8}$;

$\therefore$ (18. Cor. 1.) $8x + 32 + 8x - 24 = 9x^2 + 9x - 108$;

$\therefore$ by transposition and (17. Cor. 1.) $9x^2 - 7x = 116$,

and $x^2 - \frac{7}{9}x = \frac{116}{9}$;

completing the square,

$$x^2 - \frac{7}{9}x + \frac{49}{324} = \frac{116}{9} + \frac{49}{324} = \frac{4225}{324};$$

$\therefore$ extracting the root, $x - \frac{7}{18} = \pm\frac{65}{18}$;

$\therefore\ x = 4$, or $-\frac{29}{9}$.

22. Given $\frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3$, to find the values of x.

(18. Cor. 1.) $x^3 - 10x^2 + 1 = x^3 - 9x^2 + 27x - 27$;

$\therefore$ by transposition and (17. Cor. 1.) $x^2 + 27x = 28$;

and completing the square,

$$x^2 + 27x + \overline{\frac{27}{2}}\Big|^2 = 28 + \frac{729}{4} = \frac{841}{4};$$

$\therefore$ extracting the root, $x + \frac{27}{2} = \pm\frac{29}{2}$.

and $x = 1$, or -28.

23. Given $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}$, to find the values of x.

Clearing the equation of fractions,

$$20x^2 - 140x + 490 = 203x - 29x^2;$$

$\therefore$ by transposition, $49x^2 - 343x = -490$,

and $x^2 - 7x = -10$;

$\therefore$ completing the square, $x^2 - 7x + \frac{49}{4} = \frac{49}{4} - 10 = \frac{9}{4}$;

and extracting the root, $x - \frac{7}{2} = \pm \frac{3}{2}$;

$\therefore x = 5$, or 2.

24. Given $\frac{7-12x}{x^{\frac{3}{2}}} = \frac{x}{\sqrt{x}} - \frac{8x+110}{\sqrt{x^3}}$, to find the values of x.

Clearing the equation of fractions,

$$7 - 12x = x^2 - 8x - 110;$$

$$\therefore 117 = x^2 + 4x;$$

and completing the square, $121 = x^2 + 4x + 4$;

$\therefore$ extracting the root, $\pm 11 = x + 2$;

and therefore $x = 9$, or -13.

25. Given $\sqrt{x+5} \times \sqrt{x+12} = 12$, to find the values of x.

Squaring both sides, $\overline{x+5} \cdot \overline{x+12} = 144$,

or $x^2 + 17x + 60 = 144$;

$\therefore$ by transposition, $x^2 + 17x = 84$;

and completing the square,

$$x^2 + 17x + \left(\frac{17}{2}\right)^2 = \frac{289}{4} + 84 = \frac{625}{4};$$

extracting the root, $x+\frac{17}{2}=\pm\frac{25}{2}$.

and $x=4$, or -21.

26. Given $\sqrt[3]{x^3-a^3}=x-b$, to find the values of x.

Cubing each side, $x^3-a^3=x^3-3bx^2+3b^2x-b^3$;

$\therefore$ by transposition, $3bx^2-3b^2x=a^3-b^3$,

and $x^2-bx=\frac{a^3-b^3}{3b}$;

$\therefore$ completing the square,

$$x^2-bx+\frac{b^2}{4}=\frac{a^3-b^3}{3b}+\frac{b^2}{4}=\frac{4a^3-b^3}{12b};$$

extracting the root, $x-\frac{b}{2}=\pm\sqrt{\frac{4a^3-b^3}{12b}}$,

and $x=\frac{b}{2}\pm\sqrt{\frac{4a^3-b^3}{12b}}$.

27. Given $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$, to find the values of x.

Clearing the equation of fractions, $2x+2\sqrt{x}=16-x$;

$\therefore$ by transposition, $3x+2\sqrt{x}=16$,

and $x+\frac{2}{3}\sqrt{x}=\frac{16}{3}$;

completing the square, $x+\frac{2}{3}\sqrt{x}+\frac{1}{9}=\frac{16}{3}+\frac{1}{9}=\frac{49}{9}$;

and extracting the root, $\sqrt{x}+\frac{1}{3}=\pm\frac{7}{3}$;

$\therefore\sqrt{x}=2$, or $-\frac{8}{3}$;

and $x=4$, or $\frac{64}{9}$.

28. Given $\sqrt{x^3}-2\sqrt{x}-x=0$, to find the values of x.

Dividing by $\sqrt{x}$, we find $x-2-\sqrt{x}=0$;

$\therefore$ by transposition, $x-\sqrt{x}=2$;

and completing the square, $x-\sqrt{x}+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4}$;

extracting the root, $\sqrt{x}-\frac{1}{2}=\pm\frac{3}{2}$;

$\therefore \sqrt{x}=2$, or -1,

and $x=4$, or 1.

29. Given $\sqrt{x^5}+\sqrt{x^3}=6\sqrt{x}$, to find the values of x.

Dividing by $\sqrt{x}$, $x^2+x=6$;

$\therefore$ completing the square, $x^2+x+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

and extracting the root, $x+\frac{1}{2}=\pm\frac{5}{2}$,

and $x=2$, or -3.

30. Given $\frac{x}{2}=22\frac{1}{6}+\frac{\sqrt{x}}{3}$, to find the values of x.

Multiplying by 2, and transposing, $x-\frac{2}{3}\sqrt{x}=\frac{133}{3}$;

and completing the square,

$$x-\frac{2}{3}\sqrt{x}+\frac{1}{9}=\frac{133}{3}+\frac{1}{9}=\frac{400}{9};$$

$\therefore$ extracting the root, $\sqrt{x}-\frac{1}{3}=\pm\frac{20}{3}$,

$$\text{and } \sqrt{x}=7, \text{ or } -\frac{19}{3};$$

$$\therefore\ x=49, \text{ or } \frac{361}{9}.$$

31. Given $\dfrac{\dfrac{3\sqrt{x}}{5}-2}{x-5}-\dfrac{1}{20}=0$, to find the values of x.

Clearing the equation of fractions,

$$12\sqrt{x}-40-x+5=0;$$

$$\therefore\ (17.\ \text{Cor. } 1.)\ x-12\sqrt{x}=-35,$$

and completing the square, $x-12\sqrt{x}+36=36-35=1$;

$$\therefore \text{ extracting the root, } \sqrt{x}-6=\pm 1;$$

$$\therefore\ \sqrt{x}=7, \text{ or } 5,$$

$$\text{and } x=49, \text{ or } 25.$$

32. Given $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756$, to find the values of x.

Completing the square, $x^{\frac{6}{5}}+x^{\frac{3}{5}}+\dfrac{1}{4}=756+\dfrac{1}{4}=\dfrac{3025}{4}$;

$$\text{and extracting the root, } x^{\frac{3}{5}}+\frac{1}{2}=\pm\frac{55}{2};$$

$$\therefore\ x^{\frac{3}{5}}=27, \text{ or } -28,$$

$$\text{and } x^{\frac{1}{5}}=3, \text{ or } \sqrt[3]{-28};$$

$$\therefore\ x=243, \text{ or } \overline{-28}^{\frac{5}{3}}.$$

33. Given $x^3-x^{\frac{3}{2}}=56$, to find the values of x.

Completing the square, $x^3-x^{\frac{3}{2}}+\dfrac{1}{4}=56+\dfrac{1}{4}=\dfrac{225}{4}$;

$$\therefore \text{ extracting the root, } x^{\frac{3}{2}}-\frac{1}{2}=\pm\frac{15}{2},$$

$$\text{and } x^{\frac{3}{2}}=8, \text{ or } -7;$$

$$\therefore\ x^{\frac{1}{2}}=2, \text{ or } \sqrt[3]{-7},$$

$$\text{and } x=4, \text{ or } \overline{-7}|^{\frac{2}{3}}.$$

34. Given $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104$, to find the values of x.

$$\text{Dividing by 3, } x^{\frac{5}{3}}+\frac{1}{3}\cdot x^{\frac{5}{6}}=\frac{3104}{3};$$

and completing the square,

$$x^{\frac{5}{3}}+\frac{1}{3}x^{\frac{5}{6}}+\frac{1}{36}=\frac{3104}{3}+\frac{1}{36}=\frac{37249}{36};$$

$$\therefore \text{ extracting the root, } x^{\frac{5}{6}}+\frac{1}{6}=\pm\frac{193}{6};$$

$$\text{whence } x^{\frac{5}{6}}=32, \text{ or } -\frac{97}{3},$$

$$\text{and } x^{\frac{1}{6}}=2, \text{ or } \overline{\frac{-97}{3}}\Big|^{\frac{1}{5}};$$

$$\therefore\ x=64, \text{ or } \overline{\frac{-97}{3}}\Big|^{\frac{6}{5}}.$$

35. Given $ax^{\frac{3}{2}}+bx^{\frac{3}{4}}=c$, to find the values of x.

$$\text{Dividing by } a,\ x^{\frac{3}{2}}+\frac{b}{a}\cdot x^{\frac{3}{4}}=\frac{c}{a};$$

and completing the square,

$$x^{\frac{3}{2}}+\frac{b}{a}\cdot x^{\frac{3}{4}}+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}+\frac{c}{a}=\frac{b^2+4ac}{4a^2};$$

$$\therefore \text{ extracting the root, } x^{\frac{3}{4}}+\frac{b}{2a}=\frac{\pm\sqrt{b^2+4ac}}{2a},$$

$$\text{and } x^{\frac{3}{4}}=\frac{\pm\sqrt{b^2+4ac}-b}{2a};$$

$$\text{whence } x=\overline{\frac{\pm\sqrt{b^2+4ac}-b}{2a}}\Big|^{\frac{4}{3}}.$$

36. Given $3x^{\frac{4}{3}} - \frac{5x^{\frac{8}{3}}}{2} = -592$, to find the values of x.

$$(17.\text{ Cor. } 1.)\ \frac{5x^{\frac{8}{3}}}{2} - 3x^{\frac{4}{3}} = 592;$$

$$\text{and therefore } x^{\frac{8}{3}} - \frac{6}{5}x^{\frac{4}{3}} = \frac{1184}{5}.$$

Completing the square,

$$x^{\frac{8}{3}} - \frac{6}{5}x^{\frac{4}{3}} + \frac{9}{25} = \frac{1184}{5} + \frac{9}{25} = \frac{5929}{25};$$

$$\therefore \text{ extracting the root, } x^{\frac{4}{3}} - \frac{3}{5} = \pm\frac{77}{5};$$

$$\therefore x^{\frac{4}{3}} = 16, \text{ or } -\frac{74}{5},$$

$$\text{and } x = 8, \text{ or } \overline{-\frac{74}{5}}\Big|^{\frac{3}{4}}.$$

37. Given $x^n - 2ax^{\frac{n}{2}} = b$, to find the values of x.

$$\text{Completing the square, } x^n - 2ax^{\frac{n}{2}} + a^2 = a^2 + b;$$

$$\therefore \text{ extracting the root, } x^{\frac{n}{2}} - a = \pm\sqrt{a^2+b},$$

$$\text{and } x^{\frac{n}{2}} = a \pm \sqrt{a^2+b};$$

$$\therefore x = \overline{a \pm \sqrt{a^2+b}}\Big|^{\frac{2}{n}}.$$

38. Given $\sqrt{2x+1} + 2\sqrt{x} = \frac{21}{\sqrt{2x+1}}$, to find the values of x.

$$(18.\text{ Cor. } 1.)\ 2x + 1 + 2\sqrt{2x^2+x} = 21;$$

$$\therefore \text{ by transposition, } 2\sqrt{2x^2+x} = 20 - 2x;$$

$$\text{and therefore } \sqrt{2x^2+x} = 10 - x;$$

$\therefore$ squaring both sides, $2x^2+x=100-20x+x^2$;

and transposing, $x^2+21x=100$;

completing the square,

$$x^2+21x+\overline{\frac{21}{2}}\Big|^2=100+\frac{441}{4}=\frac{841}{4};$$

$\therefore$ extracting the root, $x+\frac{21}{2}=\pm\frac{29}{2}$;

$\therefore$ $x=4$, or -25.

39. Given

$$\sqrt{x+60}+\sqrt{x^2+9}=\frac{2\sqrt{x^3+60x^2+9x+540}+89}{\sqrt{x+60}+\sqrt{x^2+9}},$$

to find the values of x.

Clearing the equation of fractions,

$x+60+x^2+9+2\sqrt{x^3+60x^2+9x+540}=$ $2\sqrt{x^3+60x^2+9x+540}+89$;

$\therefore$ by transposition, $x^2+x=20$;

and completing the square, $x^2+x+\frac{1}{4}=20+\frac{1}{4}=\frac{81}{4}$,

$\therefore$ extracting the root, $x+\frac{1}{2}=\pm\frac{9}{2}$,

and $x=4$, or -5.

40. Given

$$\frac{123+41\sqrt{x}}{5\sqrt{x}-x}=\frac{20\sqrt{x}+4x}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x).(3-\sqrt{x})},$$

to find the values of x.

This equation is

$$\frac{41.(3+\sqrt{x})}{5\sqrt{x}-x}=\frac{4.(5\sqrt{x}+x)}{3-\sqrt{x}}-\frac{2x^2}{(5\sqrt{x}-x).(3-\sqrt{x})};$$

and therefore clearing it of fractions,

$$41\ (9-x)=4.(25x-x^2)-2x^2,$$

or $369 - 41x = 100x - 4x^2 - 2x^2$;

$\therefore$ by transposition, $6x^2 - 141x = -369$,

$$\text{and } x^2 - \frac{47}{2}x = -\frac{123}{2};$$

completing the square,

$$x^2 - \frac{47}{2}x + \overline{\frac{47}{4}}\Big|^2 = \frac{2209}{16} - \frac{123}{2} = \frac{1225}{16};$$

$$\therefore \text{ extracting the root, } x - \frac{47}{4} = \pm\frac{35}{4},$$

$$\text{and } x = \frac{41}{2}, \text{ or } 3.$$

41. Given $x + 5 = \sqrt{x+5} + 6$, to find the values of x.

By transposition, $\overline{x+5} - \sqrt{x+5} = 6$;

and therefore completing the square,

$$\overline{x+5} - \sqrt{x+5} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4};$$

$$\text{extracting the root, } \sqrt{x+5} - \frac{1}{2} = \pm\frac{5}{2};$$

$$\therefore \sqrt{x+5} = 3, \text{ or } -2;$$

and squaring both sides, $x + 5 = 9$, or 4;

whence $x = 4$, or -1.

42. Given $x + 16 - 7\sqrt{x+16} = 10 - 4\sqrt{x+16}$, to find the values of x.

By transposition, $\overline{x+16} - 3\sqrt{x+16} = 10$;

and completing the square,

$$\overline{x+16} - 3\sqrt{x+16} + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4};$$

$$\therefore \text{ extracting the root, } \sqrt{x+16} - \frac{3}{2} = \pm\frac{7}{2},$$

and $\sqrt{x+16}=5$, or -2;
whence $x+16=25$, or 4;
and $x=9$, or -12.

43. Given $\sqrt{x+12}+\sqrt[4]{x+12}=6$, to find the values of x.

Completing the square,

$$\sqrt{x+12}+\sqrt[4]{x+12}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4};$$

$\therefore$ extracting the root, $\sqrt[4]{x+12}+\frac{1}{2}=\pm\frac{5}{2}$,

and $\sqrt[4]{x+12}=2$, or -3;
whence $x+12=16$, or 81;
and $x=4$, or 69.

44. Given $x^2-2x+6\sqrt{x^2-2x+5}=11$, to find the values of x.

Adding 5 to each side of the equation,

$$\overline{x^2-2x+5}+6\sqrt{x^2-2x+5}=16;$$

$\therefore$ completing the square,

$$\overline{x^2-2x+5}+6\sqrt{x^2-2x+5}+9=16+9=25;$$

and extracting the root, $\sqrt{x^2-2x+5}+3=\pm 5$,

and $\sqrt{x^2-2x+5}=2$, or -8;
$\therefore$ squaring both sides, $x^2-2x+5=4$, or 64;
whence $x^2-2x+1=0$, or 60;
and extracting the root, $x-1=0$, or $\pm\sqrt{60}$;*
$\therefore$ $x=1$, or $1\pm 2\sqrt{15}$.

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* In this example, if 0 be the value, the two roots of the equation are $+1$ and $+1$.

45. Given $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$, to find the values of x.

Adding 6 to each side, $2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$; and completing the square,

$$\overline{2x^2+3x+9}-5\sqrt{2x^2+3x+9}+\frac{25}{4}=6+\frac{25}{4}=\frac{49}{4};$$

$\therefore$ extracting the root, $\sqrt{2x^2+3x+9}-\frac{5}{2}=\pm\frac{7}{2}$;

and $\sqrt{2x^2+3x+9}=6$, or -1;

suppose the value to be 6,

then $2x^2+3x+9=36$,

and $x^2+\frac{3}{2}.x=\frac{27}{2}$;

$\therefore$ completing the square,

$$x^2+\frac{3}{2}x+\frac{9}{16}=\frac{27}{2}+\frac{9}{16}=\frac{225}{16};$$

and extracting the root, $x+\frac{3}{4}=\pm\frac{15}{4}$,

or $x=3$, or $-\frac{9}{2}$.

But if -1 be taken, $\sqrt{2x^2+3x+9}=-1$;

$\therefore\ 2x^2+3x+9=1$,

and $x^2+\frac{3}{2}.x=-4$;

completing the square,

$$x^2+\frac{3}{2}x+\frac{9}{16}=\frac{9}{16}-4=\frac{-55}{16};$$

and extracting the root, $x+\frac{3}{4}=\frac{\pm\sqrt{-55}}{4}$;

and $x=\frac{-3\pm\sqrt{-55}}{4}$,

46. Given $\dfrac{\sqrt{x^2+x+6}}{3}=\dfrac{18-\left(\frac{4}{3}\sqrt{x^2+x+6}-2\right)}{\sqrt{x^2+x+6}}$, to find the values of x.

(18. Cor. 1.) $\overline{x^2+x+6}=54-4\sqrt{x^2+x+6}+6$;

$\therefore$ by transposition, $\overline{x^2+x+6}+4\sqrt{x^2+x+6}=60$;

and completing the square,

$$\overline{x^2+x+6}+4\sqrt{x^2+x+6}+4=64;$$

extracting the root, $\sqrt{x^2+x+6}+2=\pm 8$,

and $\sqrt{x^2+x+6}=6$, or -10; suppose the former,

then, squaring both sides, $x^2+x+6=36$;

and by transposition, $x^2+x=30$;

completing the square, $x^2+x+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4}$;

extracting the root, $x+\frac{1}{2}=\pm\frac{11}{2}$,

and $x=5$, or -6.

But taking $\sqrt{x^2+x+6}=-10$,

then $x^2+x+6=100$;

and by transposition, $x^2+x=94$;

completing the square, $x^2+x+\frac{1}{4}=94+\frac{1}{4}=\frac{377}{4}$;

extracting the root, $x+\frac{1}{2}=\frac{\pm\sqrt{377}}{2}$,

and $x=\dfrac{-1\pm\sqrt{377}}{2}$.

47. Given $\overline{\overline{x-2}^2-x}^2-\overline{x-2}^2=88-\overline{x-2}$, to find the values of x.

By transposition, $\overline{\overline{x-2}^2-x}^2-\overline{x-2}^2-x=90$;

and completing the square,

$$\overline{\overline{x-2}|^2-x}|^2-\overline{\overline{x-2}|^2-x}+\frac{1}{4}=90+\frac{1}{4}=\frac{361}{4};$$

extracting the root, $\overline{x-2}|^2-x-\frac{1}{2}=\pm\frac{19}{2}$,

and $\overline{x-2}|^2-x=10$, or -9;

whence, adding 2 to each side,

$\overline{x-2}|^2-\overline{x-2}=12$, or -7; supposing the former, and completing the square,

$$\overline{x-2}|^2-\overline{x-2}+\frac{1}{4}=12+\frac{1}{4}=\frac{49}{4};$$

extracting the root, $x-2-\frac{1}{2}=\pm\frac{7}{2}$;

whence $x=\frac{5}{2}\pm\frac{7}{2}=6$, or -1;

and in the second case, where $\overline{x-2}|^2-\overline{x-2}=-7$; completing the square,

$$\overline{x-2}|^2-\overline{x-2}+\frac{1}{4}=\frac{1}{4}-7=\frac{-27}{4};$$

extracting the root, $x-2-\frac{1}{2}=\frac{\pm\sqrt{-27}}{2}$;

$$\therefore x=\frac{5\pm3\sqrt{-3}}{2}.$$

48. Given $\overline{x+6}|^2+2x^{\frac{1}{2}}.\overline{x+6}=138+x^{\frac{1}{2}}$, to find the values of x.

Completing the square,

$$\overline{x+6}|^2+2x^{\frac{1}{2}}\times\overline{x+6}+x=138+x+x^{\frac{1}{2}};$$

extracting the root, $x+6+x^{\frac{1}{2}}=\pm\sqrt{138+x+x^{\frac{1}{2}}}$;

and squaring both sides,

$$\overline{x+x^{\frac{1}{2}}}|^2+12\,.\,\overline{x+x^{\frac{1}{2}}}+36=138+x+x^{\frac{1}{2}};$$

$\therefore$ by transposition, $\overline{x+x^{\frac{1}{2}}}|^2+11.\overline{x+x^{\frac{1}{2}}}=102$;
completing the square,

$$\overline{x+x^{\frac{1}{2}}}|^2+11.\overline{x+x^{\frac{1}{2}}}+\frac{121}{4}=102+\frac{121}{4}=\frac{529}{4};$$

extracting the root, $x+x^{\frac{1}{2}}+\frac{11}{2}=\pm\frac{23}{2}$,

and $x+x^{\frac{1}{2}}=6$, or -17; supposing the former,

and completing the square, $x+x^{\frac{1}{2}}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

extracting the root, $x^{\frac{1}{2}}+\frac{1}{2}=\pm\frac{5}{2}$,

and $x^{\frac{1}{2}}=2$, or -3;

$\therefore\ x=4$, or 9.

But if $x+x^{\frac{1}{2}}=-17$,

completing the square, $x+x^{\frac{1}{2}}+\frac{1}{4}=\frac{1}{4}-17=\frac{-67}{4}$;

and extracting the root, $x^{\frac{1}{2}}+\frac{1}{2}=\frac{\pm\sqrt{-67}}{2}$,

and $x^{\frac{1}{2}}=\frac{-1\pm\sqrt{-67}}{2}$;

whence $x=\frac{-33\mp\sqrt{-67}}{2}$.

49. Given $x-1=2+\frac{2}{x^{\frac{1}{2}}}$, to find the values of x.

Since $x-1=(x^{\frac{1}{2}}+1)\times(x^{\frac{1}{2}}-1)$;

$\therefore\ (x^{\frac{1}{2}}+1).(x^{\frac{1}{2}}-1)=2.\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}}$;

and therefore, dividing by $x^{\frac{1}{2}}+1$, $x^{\frac{1}{2}}-1=\frac{2}{x^{\frac{1}{2}}}$;

whence $x-x^{\frac{1}{2}}=2$;

and completing the square, $x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$;

$\therefore$ extracting the root, $x^{\frac{1}{2}} - \frac{1}{2} = \pm\frac{3}{2}$,

and $x^{\frac{1}{2}} = 2$, or -1;

whence $x = 4$, or 1.

50. Given $x^4 - 2x^3 + x = 132$, to find the values of x.

(15) Adding and subtracting x^2, there results

$$x^4 - 2x^3 + x^2 - \overline{x^2 - x} = 132;$$

and completing the square,

$$\overline{x^2 - x}|^2 - \overline{x^2 - x} + \frac{1}{4} = 132 + \frac{1}{4} = \frac{529}{4};$$

extracting the root, $x^2 - x - \frac{1}{2} = \pm\frac{23}{2}$;

whence $x^2 - x = 12$, or -11; supposing the former;

then, completing the square, $x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

and extracting the root, $x - \frac{1}{2} = \pm\frac{7}{2}$,

and $x = 4$, or -3.

But if $x^2 - x = -11$,

completing the square, $x^2 - x + \frac{1}{4} = \frac{1}{4} - 11 = -\frac{43}{4}$;

extracting the root, $x - \frac{1}{2} = \frac{\pm\sqrt{-43}}{2}$,

and $x = \frac{1 \pm \sqrt{-43}}{2}$.

51. Given $\overline{x^2 + 2x} \cdot \overline{x + 4} = 2 - \overline{x + 4}$, to find the values of x.

By subtraction, $\overline{x^2+2x}.\overline{x+4}=-\overline{x+2}$;

$\therefore$ dividing by $x+2$, $x.\overline{x+4}=-1$;

$\therefore x^2+4x=-1$;

and completing the square, $x^2+4x+4=4-1=3$;

extracting the root, $x+2=\pm\sqrt{3}$;

$\therefore x=-2\pm\sqrt{3}$.

52. Given $\dfrac{33\frac{24}{25}}{\sqrt{5x^2-x^4}}+\dfrac{\sqrt{5-x^2}}{25x}=\dfrac{34}{x}$, to find the values of x.

Multiplying every term by $25\sqrt{5x^2-x^4}$;

$\therefore 849+5-x^2=850\sqrt{5-x^2}$;

and by transposition, $\overline{5-x^2}-850\sqrt{5-x^2}=-849$;

completing the square,

$\left|\overline{5-x^2}-850\sqrt{5-x^2}+425\right|^2=180625-849=179776$;

and extracting the root, $\sqrt{5-x^2}-425=\pm 424$;

whence $\sqrt{5-x^2}=849$, or 1,

and $5-x^2=720801$, or 1,

and $x^2=-720796$, or 4;

$\therefore x=\pm\sqrt{-720796}$, or ± 2.

53. Given $9x+\sqrt{16x^2+36x^3}=15x^2-4$, to find the values of x.

By transposition, $9x+4+2x\sqrt{9x+4}=15x^2$;

and completing the square,

$\overline{9x+4}+2x\sqrt{9x+4}+x^2=16x^2$;

and extracting the root, $\sqrt{9x+4}+x=\pm 4x$;

$\therefore \sqrt{9x+4}=3x$, or $-5x$,

and $9x+4=9x^2$, or $25x^2$; supposing the former;

then, by transposition, $9x^2-9x=4$;

and completing the square, $9x^2-9x+\frac{9}{4}=4+\frac{9}{4}=\frac{25}{4}$;

$\therefore$ extracting the root, $3x-\frac{3}{2}=\pm\frac{5}{2}$;

and $3x=4$, or -1;

whence $x=\frac{4}{3}$, or $-\frac{1}{3}$.

But if $9x+4=25x^2$,

then, by transposition, $25x^2-9x=4$;

and completing the square,

$$25x^2-9x+\overline{\frac{9}{10}}\Big|^2=\frac{81}{100}+4=\frac{481}{100};$$

and extracting the root, $5x-\frac{9}{10}=\pm\frac{\sqrt{481}}{10}$;

$$\therefore\ 5x=\frac{9\pm\sqrt{481}}{10},$$

$$\text{and } x=\frac{9\pm\sqrt{481}}{50}.$$

54. Given $x=\frac{12+8x^{\frac{1}{2}}}{x-5}$, to find the values of x.

(18. Cor. 1.) $x^2-5x=12+8x^{\frac{1}{2}}$,

or $x^2-4x=12+8x^{\frac{1}{2}}+x$;

and completing the square, $x^2-4x+4=16+8x^{\frac{1}{2}}+x$;

extracting the root, $x-2=\pm\ (4+x^{\frac{1}{2}})$, and first taking the positive value;

then, by transposition, $x-x^{\frac{1}{2}}=6$;

completing the square, $x-x^{\frac{1}{2}}+\frac{1}{4}=6+\frac{1}{4}=\frac{25}{4}$;

extracting the root, $x^{\frac{1}{2}}-\frac{1}{2}=\pm\frac{5}{2}$;

$$\therefore\ x^{\frac{1}{2}}=3, \text{ or } -2,$$

$$\text{and } x=9, \text{ or } 4.$$

But if the negative value be used, $x-2=-4-x^{\frac{1}{2}}$;

$\therefore$ by transposition $x+x^{\frac{1}{2}}=-2$;

and completing the square, $x+x^{\frac{1}{2}}+\frac{1}{4}=\frac{1}{4}-2=\frac{-7}{4}$;

extracting the root, $x^{\frac{1}{2}}+\frac{1}{2}=\frac{\pm\sqrt{-7}}{2}$;

$$\therefore\ x^{\frac{1}{2}}=\frac{-1\pm\sqrt{-7}}{2},$$

$$\text{and } x=\frac{-3\mp\sqrt{-7}}{2}.$$

55. Given $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$, to find the values of x.

Adding $\frac{1}{x^2}$ to each side, in order to complete the square,

then $\frac{49x^2}{4}-49+\frac{49}{x^2}=9+\frac{6}{x}+\frac{1}{x^2}$;

extracting the root, $\frac{7x}{2}-\frac{7}{x}=\pm\left(3+\frac{1}{x}\right)$,

and first taking the positive value;

$\therefore$ by transposition, $\frac{7x}{2}-3=\frac{8}{x}$;

and therefore $x^2-\frac{6x}{7}=\frac{16}{7}$;

completing the square,

$$x^2-\frac{6x}{7}+\frac{9}{49}=\frac{16}{7}+\frac{9}{49}=\frac{121}{49};$$

$$\text{extracting the root, } x-\frac{3}{7}=\pm\frac{11}{7},$$

$$\text{and } x=2, \text{ or } -\frac{8}{7}.$$

But if the negative value be used, $\frac{7x}{2}-\frac{7}{x}=-3-\frac{1}{x}$;

$$\therefore \text{ by transposition, } \frac{7x}{2}+3=\frac{6}{x},$$

$$\text{and } x^2+\frac{6x}{7}=\frac{12}{7};$$

$\therefore$ completing the square,

$$x^2+\frac{6x}{7}+\frac{9}{49}=\frac{12}{7}+\frac{9}{49}=\frac{93}{49};$$

$$\text{and extracting the root, } x+\frac{3}{7}=\frac{\pm\sqrt{93}}{7};$$

$$\therefore x=\frac{-3\pm\sqrt{93}}{7}.$$

56. Given $\frac{x^4}{2}+\frac{17x^3}{4}-17x=8$, to find the values of x.

$$\text{Multiplying by 2, } x^4+\frac{17x^3}{2}-34x=16;$$

$$\therefore \text{ by transposition, } x^4+\frac{17x^3}{2}=34x+16;$$

and completing the square,

$$x^4+\frac{17x^3}{2}+\overline{\frac{17x}{4}}\Big|^2=\overline{\frac{17x}{4}}\Big|^2+34x+16;$$

$$\text{extracting the root, } x^2+\frac{17x}{4}=\pm\left(\frac{17x}{4}+4\right);$$

first let the positive value be taken;

$$\text{then, by transposition, } x^2=4,$$

$$\text{and } x=\pm 2.$$

But if the negative value be taken,

$$x^2 + \frac{17x}{4} = -\frac{17x}{4} - 4;$$

and by transposition, $x^2 + \frac{17x}{2} = -4;$

∴ completing the square,

$$x^2 + \frac{17x}{2} + \overline{\frac{17}{4}}\Big|^2 = \frac{289}{16} - 4 = \frac{225}{16};$$

and extracting the root, $x + \frac{17}{4} = \pm\frac{15}{4};$

$$\therefore\ x = -8, \text{ or } -\frac{1}{2}.$$

57. Given $27x^2 - \frac{841}{3x^2} + \frac{17}{3} = \frac{232}{3x} - \frac{1}{3x^2} + 5,$ to find the values of x.

Multiplying every term by 3,

$$81x^2 - \frac{841}{x^2} + 17 = \frac{232}{x} - \frac{1}{x^2} + 15;$$

∴ by transposition, $81x^2 + 17 + \frac{1}{x^2} = \frac{841}{x^2} + \frac{232}{x} + 15.$

Adding unity to each side, in order to complete the square;

$$\therefore\ 81x^2 + 18 + \frac{1}{x^2} = \frac{841}{x^2} + \frac{232}{x} + 16;$$

and extracting the root, $9x + \frac{1}{x} = \pm\left(\frac{29}{x} + 4\right).$

Let the positive value be taken;

then, by transposition, $9x - 4 = \frac{28}{x};$

and therefore $9x^2 - 4x = 28;$

completing the square, $9x^2 - 4x + \frac{4}{9} = \frac{4}{9} + 28 = \frac{256}{9};$

$$\text{extracting the root, } 3x - \frac{2}{3} = \pm\frac{16}{3};$$

$$\therefore\ 3x = 6, \text{ or } -\frac{14}{3},$$

$$\text{and } x = 2, \text{ or } -\frac{14}{9}.$$

But if the negative value be taken, $9x^2 + 4x = -30$; and completing the square,

$$9x^2 + 4x + \frac{4}{9} = \frac{4}{9} - 30 = \frac{-266}{9};$$

$$\text{extracting the root, } 3x + \frac{2}{3} = \frac{\pm\sqrt{-266}}{3};$$

$$\therefore\ 3x = \frac{-2 \pm \sqrt{-266}}{3},$$

$$\text{and } x = \frac{-2 \pm \sqrt{-266}}{9}.$$

58. Given $\overline{x^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}} + \overline{a^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}} = \frac{x^2}{a}$, to find the values of x.

$$\text{By transposition, } \overline{x^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}} - \frac{x^2}{a} = -\overline{a^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}};$$

and squaring both sides,

$$x^2 - \frac{a^4}{x^2} + \frac{x^4}{a^2} - \frac{2x^2}{a} \cdot \overline{x^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}} = a^2 - \frac{a^4}{x^2};$$

$$\text{and by transposition, } x^2 - \frac{2x^2}{a} \cdot \overline{x^2 - \frac{a^4}{x^2}}\Big|^{\frac{1}{2}} + \frac{x^4}{a^2} - a^2 = 0,$$

$$\text{or } x^2 - \frac{2x}{a} \cdot \overline{x^4 - a^4}\Big|^{\frac{1}{2}} + \frac{x^4 - a^4}{a^2} = 0;$$

$$\text{extracting the root, } x - \frac{\sqrt{x^4 - a^4}}{a} = 0;$$

$\therefore$ by transposition, $x = \frac{\sqrt{x^4 - a^4}}{a}$,

and $ax = \sqrt{x^4 - a^4}$.

Squaring both sides, $a^2x^2 = x^4 - a^4$;

$\therefore$ by transposition, $x^4 - a^2x^2 = a^4$;

and completing the square,

$$x^4 - a^2x^2 + \frac{a^4}{4} = a^4 + \frac{a^4}{4} = \frac{5a^4}{4};$$

extracting the root, $x^2 - \frac{a^2}{2} = \frac{\pm\sqrt{5}\,.\,a^2}{2}$;

$$\therefore\ x^2 = \frac{1 \pm \sqrt{5}}{2}\,.\,a^2,$$

and $x = \pm a\sqrt{\frac{1 \pm \sqrt{5}}{2}}$.

Sect. V.

Solution of Adfected Quadratics, involving two unknown Quantities.

(29.) If the equations involve two unknown quantities, they may, by the preceding rules, be reduced to one containing only one of the unknown quanties, the values of which may be found by Art. 28.; whence, by substitution, the values of the other may also be determined. In many cases, however, it may be convenient to solve the equations first, considering one of the quantities as known; when the rules for exterminating unknown quantities (23) may be more easily applied.

Examples.

1. Given $x-y=15$ and $\frac{xy}{2}=y^3$, to find the values of x and y.

From the second equation, $x=2y^2$;
Substituting this in the first, $2y^2-y=15$;

$$\therefore y^2-\frac{1}{2}y=\frac{15}{2};$$

and completing the square,

$$y^2-\frac{1}{2}y+\frac{1}{16}=\frac{15}{2}+\frac{1}{16}=\frac{121}{16};$$

$\therefore$ extracting the root, $y - \frac{1}{4} = \pm \frac{11}{4}$,

and $y = 3$, or $-\frac{5}{2}$;

whence $x = 2y^2 = 18$, or $\frac{25}{2}$.

2. Given $\left.\begin{array}{l} \frac{10x+y}{xy} = 3 \\ \text{and } 9y - 9x = 18 \end{array}\right\}$; to find the values of x and y.

From the second equation, $y - x = 2$;

and therefore $y = x + 2$;

but from the first, $10x + y = 3xy$.

Substituting in this the value of y found above,

$$10x + x + 2 = 3x \cdot (x + 2),$$

or $11x + 2 = 3x^2 + 6x$;

$\therefore$ by transposition, $3x^2 - 5x = 2$,

and (18) $x^2 - \frac{5}{3}x = \frac{2}{3}$;

$\therefore$ completing the square,

$$x^2 - \frac{5}{3} \cdot x + \frac{25}{36} = \frac{25}{36} + \frac{2}{3} = \frac{49}{36};$$

$\therefore$ extracting the root, $x - \frac{5}{6} = \pm \frac{7}{6}$,

and $x = 2$, or $-\frac{1}{3}$;

whence $y = x + 2 = 4$, or $\frac{5}{3}$.

3. Given $\left.\begin{array}{l} x+y : x-y :: 13 : 5 \\ \text{and } y^2 + x = 25 \end{array}\right\}$, to find the values of x and y.

$$(Alg.\ 182.)\ 2x : 2y :: 18 : 8$$
$$\therefore (Alg.\ 184.)\ x : y :: 9 : 4;$$
$$\therefore (21)\ 4x = 9y, \text{ and } x = \frac{9y}{4}.$$

Substituting this value in the second equation,

$$y^2 + \frac{9y}{4} = 25;$$

completing the square,

$$y^2 + \frac{9y}{4} + \frac{81}{64} = \frac{81}{64} + 25 = \frac{1681}{64};$$
$$\therefore \text{ extracting the root, } y + \frac{9}{8} = \pm\frac{41}{8};$$
$$\text{whence } y = 4, \text{ or } -\frac{25}{4};$$
$$\therefore x = \frac{9y}{4} = 9, \text{ or } -\frac{225}{16}.$$

4. Given $\left.\begin{array}{l} 4xy = 96 - x^2y^2 \\ \text{and } x + y = 6 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^2y^2 + 4xy = 96$;
$\therefore$ completing the square, $x^2y^2 + 4xy + 4 = 100$;
and extracting the root, $xy + 2 = \pm 10$;
$\therefore xy = 8$, or -12.

Now, squaring the second equation,

$$x^2 + 2xy + y^2 = 36;$$
$$\text{but } 4xy = 32, \text{ or } -48;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 4, \text{ or } 84;$$

whence, extracting the root, $x - y = \pm 2$, or $\pm\sqrt{84}$;
but $x + y = 6$;

$$\therefore \text{ by addition, } 2x = 8, \text{ or } 4, \text{ or } 6 \pm 2\sqrt{21};$$

whence $x=4$, or 2, or $3\pm\sqrt{21}$;
and by subtraction, $2y=4$, or 8, or $6\mp 2\sqrt{21}$;
$\therefore\ y=2$, or 4, or $3\mp\sqrt{21}$.

5. Given $\left.\begin{array}{l} x^2+x+y=18-y^2 \\ \text{and } xy=6 \end{array}\right\}$, to find the values of x and y.

By transposition, $x^2+y^2+x+y=18$;
and from the second equation, $2xy=12$;

$\therefore$ by addition, $x^2+2xy+y^2+x+y=30$;
and completing the square,

$$\overline{x+y}|^2+(x+y)+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4};$$

$\therefore$ extracting the root, $x+y+\frac{1}{2}=\pm\frac{11}{2}$,
and $x+y=5$, or -6;
whence, from the first equation, $x^2+y^2=13$, or 24;
but $2xy=12$;

$\therefore$ by subtraction, $x^2-2xy+y^2=1$, or 12;
$\therefore\ x-y=\pm 1$, or $\pm 2\sqrt{3}$.
Now $x+y=5$, or -6;

$\therefore$ by addition, $2x=6$, or 4, or $-6\pm 2\sqrt{3}$;
$\therefore\ x=3$, or 2, or $-3\pm\sqrt{3}$;
and by subtraction, $2y=4$, or 6, or $-6\mp 2\sqrt{3}$;
$\therefore\ y=2$, or 3, or $-3\mp\sqrt{3}$.

6. Given $\left.\begin{array}{l} x^2+2xy+y^2+2x=120-2y \\ \text{and } xy-y^2=8 \end{array}\right\}$, to find the values of x and y.

By transposition, $\overline{x+y}|^2+2.(x+y)=120$;

$\therefore$ completing the square, $\overline{x+y}|^2+2.(x+y)+1=121$;

$\therefore$ extracting the root, $(x+y)+1=\pm 11$,

and $x+y=10$, or -12; and first let $x+y=10$;

from the second equation, $x-y=\frac{8}{y}$;

$\therefore$ by subtraction, $2y=10-\frac{8}{y}$;

$\therefore y^2=5y-4$;

and by transposition, $y^2-5y=-4$;

$\therefore$ completing the square, $y^2-5y+\frac{25}{4}=\frac{25}{4}-4=\frac{9}{4}$;

and extracting the root, $y-\frac{5}{2}=\pm\frac{3}{2}$;

$\therefore y=4$, or 1;

and $x=10-y=6$, or 9.

But if $x+y=-12$,

and $x-y=\frac{8}{y}$;

then $2y=-12-\frac{8}{y}$,

and $y^2+6y=-4$;

$\therefore$ completing the square, $y^2+6y+9=9-4=5$;

extracting the root, $y+3=\pm\sqrt{5}$;

$\therefore y=-3\pm\sqrt{5}$,

and $x=-12-y=-9\mp\sqrt{5}$.

7. Given $\left.\begin{array}{l} x^2+y^2-x-y=78 \\ \text{and } xy+x+y=39 \end{array}\right\}$, to find the values of x and y.

Since $x^2+y^2-(x+y)=78$;

and from the second, $2xy+2.(x+y)=78$;

$\therefore$ by addition, $x^2+2xy+y^2+x+y=156$;

and completing the square,

$$\overline{x+y}|^2+(x+y)+\frac{1}{4}=156+\frac{1}{4}=\frac{625}{4};$$

extracting the root, $x+y+\frac{1}{2}=\pm\frac{25}{2}$;

$\therefore$ $x+y=12$, or -13; supposing the former;

then $xy=39-(x+y)=39-12=27$,

and $x^2+y^2=78+(x+y)=78+12=90$;

but $2xy = 54$;

$\therefore$ by subtraction, $x^2-2xy+y^2=36$;

and extracting the root, $x-y=\pm 6$;

but $x+y=12$;

$\therefore$ by addition, $2x = 18$, or 6,

and $x = 9$, or 3;

and by subtraction, $2y = 6$, or 18,

and $y = 3$, or 9.

But if $x+y=-13$,

then $xy=39+13=52$,

and $x^2+y^2=78-13=65$;

but $2xy =104$;

$\therefore$ by subtraction, $x^2-2xy+y^2=-39$;

and extracting the root, $x-y=\pm\sqrt{-39}$;

but $x+y=-13$;

$\therefore$ by addition, $2x =-13\pm\sqrt{-39}$,

and $x =\frac{-13\pm\sqrt{-39}}{2}$;

but by subtraction, $2y = -13 \mp \sqrt{-39}$;

$$\therefore\ y = \frac{-13 \mp \sqrt{-39}}{2}.$$

8. Given $\left.\begin{array}{l} x^2y^4 - 7xy^2 - 945 = 765 \\ \text{and } xy - y = 12 \end{array}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$x^2y^4 - 7xy^2 = 1710;$$

and completing the square,

$$x^2y^4 - 7xy^2 + \frac{49}{4} = 1710 + \frac{49}{4} = \frac{6889}{4};$$

extracting the root, $xy^2 - \frac{7}{2} = \pm\frac{83}{2}$;

$$\therefore\ xy^2 = 45, \text{ or } -38.$$

Multiplying the second equation by y, $xy^2 - y^2 = 12y$.

Substituting in this the value of xy^2 found above,

$45 - y^2 = 12y$, in one case;

and by transposition, $y^2 + 12y = 45$;

completing the square, $y^2 + 12y + 36 = 45 + 36 = 81$;

extracting the root, $y + 6 = \pm 9$,

and $y = 3$, or -15;

whence $x = \frac{45}{y^2} = 5$, or $\frac{1}{5}$;

and in the other case, $-38 - y^2 = 12y$;

whence $y^2 + 12y = -38$;

completing the square, $y^2 + 12y + 36 = 36 - 38 = -2$;

extracting the root, $y + 6 = \pm\sqrt{-2}$;

$$\therefore\ y = -6 \pm \sqrt{-2},$$

and $x = \frac{-38}{y^2} = \frac{-38}{34 \mp 12\sqrt{-2}} = \frac{-19}{17 \mp 6\sqrt{-2}}$.

9. Given $\left.\begin{array}{l} x-2\sqrt{xy}+y-\sqrt{x}+\sqrt{y}=0 \\ \text{and } \sqrt{x}+\sqrt{y}=5 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$\overline{\sqrt{x}-\sqrt{y}}|^2-(\sqrt{x}-\sqrt{y})+\frac{1}{4}=\frac{1}{4};$$

and extracting the root, $\sqrt{x}-\sqrt{y}-\frac{1}{2}=\pm\frac{1}{2};$

$$\therefore \sqrt{x}-\sqrt{y}=1, \text{ or } 0;$$

but from the second equation, $\sqrt{x}+\sqrt{y}=5;$

$$\therefore \text{ by addition, } 2\sqrt{x}=6, \text{ or } 5,$$

$$\text{and } \sqrt{x}=3, \text{ or } \frac{5}{2};$$

$$\therefore x=9, \text{ or } \frac{25}{4};$$

$$\text{but by subtraction, } 2\sqrt{y}=4, \text{ or } 5,$$

$$\text{or } \sqrt{y}=2, \text{ or } \frac{5}{2};$$

$$\text{and } y=4, \text{ or } \frac{25}{4}.$$

10. Given $\left.\begin{array}{l} \frac{x^2}{y^2}+\frac{4x}{y}=\frac{85}{9} \\ \text{and } x-y=2 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$\frac{x^2}{y^2}+\frac{4x}{y}+4=\frac{85}{9}+4=\frac{121}{9};$$

and extracting the root, $\frac{x}{y}+2=\pm\frac{11}{3};$

$$\therefore \frac{x}{y}=\frac{5}{3}, \text{ or } -\frac{17}{3},$$

and $x=\frac{5y}{3}$, or $\frac{-17y}{3}$; supposing the former;

then, from the second equation, $\frac{5y}{3}-y=2,$

$$\text{or } 2y=6,$$
$$\text{and } y=3;$$
$$\therefore\ x=\frac{5y}{3}=5,$$

and if the second value be taken, $\frac{-17y}{3}-y=2$

$$\text{or } -20y=6;$$
$$\therefore\ y=\frac{-3}{10};$$
$$\text{and } x=\frac{-17y}{3}=\frac{17}{10}.$$

11. Given $\left.\begin{array}{l}\sqrt{\frac{3x}{x+y}}+\sqrt{\frac{x+y}{3x}}=2\\ \text{and } xy-(x+y)=54\end{array}\right\}$, to find the values of x and y.

(18. Cor. 1.) and by transposition,

$$3x-2\sqrt{3x}.\sqrt{x+y}+(x+y)=0;$$

$\therefore$ extracting the root, $\sqrt{3x}-\sqrt{x+y}=0$;

by transposition, $\sqrt{3x}=\sqrt{x+y}$,

and squaring both sides, $3x=x+y$,

$$\text{and } \therefore\ 2x=y;$$

substituting this value in the second equation, $2x^2-3x=54$,

$$\text{or } x^2-\frac{3}{2}.x=27;$$

whence completing the square,

$$x^2-\frac{3}{2}x+\frac{9}{16}=27+\frac{9}{16}=\frac{441}{16};$$

$$\text{extracting the root, } x - \frac{3}{4} = \pm \frac{21}{4},$$

$$\text{and } x = 6, \text{ or } -\frac{9}{2};$$

$$\text{whence } y = 2x = 12, \text{ or } -9.$$

12. Given $\left.\begin{array}{l} x^4 - 2x^2y + y^2 = 49, \\ \text{and } x^4 - 2x^2y^2 + y^4 - x^2 + y^2 = 20 \end{array}\right\}$, to find the values of x and y.

Completing the square in the second equation,

$$\overline{x^2 - y^2}|^2 - (x^2 - y^2) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

$$\text{extracting the root, } x^2 - y^2 - \frac{1}{2} = \pm \frac{9}{2};$$

$$\therefore\ x^2 - y^2 = 5, \text{ or } -4;$$

but extracting the root of the first equation, $\Big\}$; $x^2 - y = \pm 7$;

$$\therefore \text{ by subtraction, } y^2 - y = 2, \text{ or } -12, \text{ or } 11, \text{ or } -3.$$

Taking the first value, and completing the square,

$$y^2 - y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \pm \frac{3}{2},$$

$$\text{and } y = 2, \text{ or } -1;$$

$$\therefore\ x = \pm\sqrt{7 + y} = \pm 3, \text{ or } \pm\sqrt{6}.$$

Taking the second value, $y^2 - y = -12$;

$$\text{completing the square, } y^2 - y + \frac{1}{4} = \frac{1}{4} - 12 = \frac{-47}{4};$$

$$\text{extracting the root, } y - \frac{1}{2} = \frac{\pm\sqrt{-47}}{2},$$

$$\text{and } y = \frac{1 \pm \sqrt{-47}}{2};$$

$$\text{whence } x = \pm\sqrt{y-7} = \pm\sqrt{\frac{1 \pm \sqrt{-47}}{2} - 7}$$

$$= \pm\sqrt{\frac{-13 \pm \sqrt{-47}}{2}}.$$

Taking the third value, $y^2 - y = 11$;

completing the square, $y^2 - y + \frac{1}{4} = 11 + \frac{1}{4} = \frac{45}{4}$;

extracting the root, $y - \frac{1}{2} = \frac{\pm\sqrt{45}}{2}$;

$$\therefore y = \frac{1 \pm 3\sqrt{5}}{2};$$

and

$$x = \pm\sqrt{7+y} = \pm\sqrt{7 + \frac{1 \pm 3\sqrt{5}}{2}} = \pm\sqrt{\frac{15 \pm 3\sqrt{5}}{2}}.$$

Taking the fourth value, $y^2 - y = -3$;

completing the square, $y^2 - y + \frac{1}{4} = \frac{1}{4} - 3 = \frac{-11}{4}$;

extracting the root, $y - \frac{1}{2} = \frac{\pm\sqrt{-11}}{2}$,

$$\text{and } y = \frac{1 \pm \sqrt{-11}}{2};$$

$$\therefore x = \pm\sqrt{y-7}$$

$$= \pm\sqrt{\frac{1 \pm \sqrt{-11}}{2} - 7} = \pm\sqrt{\frac{-13 \pm \sqrt{-11}}{2}}.$$

13. Given $\left.\begin{array}{l} xy + xy^2 = 12, \\ \text{and } x + xy^3 = 18 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x = \dfrac{12}{y.(1+y)}$,

and from the second, $x = \frac{18}{1+y^3}$;

whence $\frac{12}{y.(1+y)} = \frac{18}{1+y^3}$, and dividing by $\frac{6}{1+y}$,

$$\frac{2}{y} = \frac{3}{y^2 - y + 1};$$

$$\therefore\ 2y^2 - 2y + 2 = 3y,$$

and by transposition, $2y^2 - 5y = -2$,

$$\text{or } y^2 - \frac{5}{2}.y = -1;$$

completing the square, $y^2 - \frac{5}{2}.y + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16}$;

extracting the root, $y - \frac{5}{4} = \pm\frac{3}{4}$;

$$\therefore\ y = 2, \text{ or } \frac{1}{2};$$

hence $x = \frac{12}{y + y^2} = 2$, or 16.

14. Given $\left.\begin{array}{l} x - x^{\frac{1}{2}} = 3 - y \\ \text{and } 4 - x = y - y^{\frac{1}{2}} \end{array}\right\}$, to find the values of x and y.

Adding the two equations together, $4 - x^{\frac{1}{2}} = 3 - y^{\frac{1}{2}}$;

$\therefore$ by transposition, $y^{\frac{1}{2}} = x^{\frac{1}{2}} - 1$,

and $y = \overline{x^{\frac{1}{2}} - 1}|^2$.

Substituting this value in the first equation,

$$\therefore\ x - x^{\frac{1}{2}} = 3 - x + 2x^{\frac{1}{2}} - 1;$$

$\therefore$ by transposition, $2x - 3x^{\frac{1}{2}} = 2$,

and $x - \frac{3}{2}.x^{\frac{1}{2}} = 1$;

completing the square, $x - \frac{3}{2}x^{\frac{1}{2}} + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16}$;

extracting the root, $x^{\frac{1}{2}} - \frac{3}{4} = \pm\frac{5}{4}$,

and $x^{\frac{1}{2}} = 2$, or $-\frac{1}{2}$;

$\therefore\ x = 4$, or $\frac{1}{4}$,

and $y = \overline{x^{\frac{1}{2}} - 1}\big|^2 = 1$, or $\frac{9}{4}$.

15. Given $\left.\begin{array}{l}\overline{x^2 + 1}\,.\,y = xy + 126 \\ \text{and } \overline{x^2+1}\,.\,y = x^2y^2 - 744\end{array}\right\}$, to find the values of x and y.

Since quantities which are equal to the same, are equal to each other,

$$x^2y^2 - 744 = xy + 126;$$

$\therefore$ by transposition, $x^2y^2 - xy = 870$;

completing the square, $x^2y^2 - xy + \frac{1}{4} = 870 + \frac{1}{4} = \frac{3481}{4}$;

extracting the root, $xy - \frac{1}{2} = \pm\frac{59}{2}$,

and $xy = 30$, or -29; let the former value be taken,

then from the first equation $(x^2 + 1).\frac{30}{x} = 156$;

$$\therefore\ x^2 + 1 = \frac{156x}{30} = \frac{26x}{5};$$

and by transposition, $x^2 - \frac{26x}{5} = -1$;

completing the square, $x^2 - \frac{26}{5}.x + \frac{169}{25} = \frac{169}{25} - 1 = \frac{144}{25}$;

extracting the root, $x - \frac{13}{5} = \pm\frac{12}{5}$;

whence $x = 5$, or $\frac{1}{5}$;

$$\text{and } \therefore\ y = \frac{30}{x} = 6, \text{ or } 150.$$

$$\text{In the second case, } \overline{x^2+1} \times -\frac{29}{x} = -29 + 126 = 97,$$

$$\text{and } x^2+1 = -\frac{97x}{29};$$

$$\text{by transposition, } x^2 + \frac{97x}{29} = -1;$$

$$\text{completing the square, } x^2 + \frac{97}{29}.x + \overline{\frac{97}{58}}\Big|^2 = \frac{9409}{3364} - 1 = \frac{6045}{3364};$$

$$\text{extracting the root, } x + \frac{97}{58} = \pm\frac{\sqrt{6045}}{58};$$

$$\therefore\ x = \frac{-97 \pm \sqrt{6045}}{58};$$

$$\therefore\ y = -\frac{29}{x} = \frac{1682}{97 \mp \sqrt{6045}}.$$

16. Given $\left.\begin{array}{l} x+y+\sqrt{x+y}=12 \\ \text{and } x^3+y^3 \quad =189 \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$x+y+\sqrt{x+y}+\frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4};$$

$$\text{extracting the root, } \sqrt{x+y} + \frac{1}{2} = \pm\frac{7}{2};$$

$$\therefore\ \sqrt{x+y} = 3, \text{ or } -4,$$

$$\text{and } x+y = 9, \text{ or } 16;$$

$$\therefore\ x^3+3x^2y+3xy^2+y^3 = 729, \text{ or } 4096;$$

$$\text{but } x^3 \qquad\qquad +y^3 = 189;$$

$$\therefore\ \text{by subtraction, } \quad 3x^2y+3xy^2 = 540, \text{ or } 3907;$$

$$\therefore\ (x+y).\ xy = 180, \text{ or } \frac{3907}{3};$$

$\therefore\ 9xy = 180$ in one case, and $16xy = \frac{3907}{3}$ in the other,

whence in the first case $xy = 20$.

$$\begin{aligned} \text{Now } x^2 + 2xy + y^2 &= 81; \\ \text{but } 4xy &= 80; \\ \therefore \text{ by subtraction, } x^2 - 2xy + y^2 &= 1; \\ \text{and extracting the root, } x - y &= \pm 1; \\ \text{but, } x + y &= 9; \\ \therefore \text{ by addition, } 2x &= 10, \text{ or } 8, \\ \text{and } x &= 5, \text{ or } 4; \\ \text{but by subtraction, } 2y &= 8, \text{ or } 10; \\ \text{and } y &= 4, \text{ or } 5. \end{aligned}$$

Now in the second case, $16xy = \frac{3907}{3}$;

$$\begin{aligned} \therefore\ 4xy &= \frac{3907}{12}; \\ \text{and since } x^2 + 2xy + y^2 &= 256, \\ \text{and } 4xy &= \frac{3907}{12}; \\ \therefore \text{ by subtraction, } x^2 - 2xy + y^2 &= -\frac{835}{12}; \\ \text{and extracting the root, } x - y &= \pm\sqrt{\frac{-835}{12}}, \\ \text{but } x + y &= 16; \\ \therefore \text{ by addition, } 2x &= 16 \pm \sqrt{\frac{-835}{12}}, \\ \text{and } x &= 8 \pm \frac{1}{4}\sqrt{\frac{-835}{3}}; \\ \text{but by subtraction, } 2y &= 16 \mp \sqrt{\frac{-835}{12}}, \\ \text{and } y &= 8 \mp \frac{1}{4}\sqrt{\frac{-835}{3}}. \end{aligned}$$

17. Given $\left.\begin{array}{l} x^2+y^2+x-y=132. \\ \text{and } (x^2+y^2).(x-y)=1220 \end{array}\right\}$, to find the values of x and y.

From the first equation $x^2+y^2=132-(x-y)$;

$$\text{and from the second } x^2+y^2=\frac{1220}{x-y};$$

$$\text{whence } 132-(x-y)=\frac{1220}{x-y};$$

$$\text{and, } \therefore\ 132.(x-y)-\overline{x-y}|^2=1220;$$

and (17. Cor. 1.) $\overline{x-y}|^2-132.(x-y)=-1220$;

completing the square,

$$\overline{x-y}|^2-132.(x-y)+\overline{66}|^2=4356-1220=3136;$$

$$\text{extracting the root, } x-y-66=\pm 56;$$

and $x-y=10$, or 122, supposing the former,

$$\begin{array}{rr} & x^2+y^2=122; \\ \text{but} & x^2-2xy+y^2=100; \\ \hline \therefore \text{ by subtraction,} & 2xy=\ 22, \\ \text{and since} & x^2+\quad y^2=122; \\ \hline \therefore \text{ by addition,} & x^2+2xy+y^2=144, \end{array}$$

$$\text{and extracting the root, } x+y=\pm 12;$$

$$\begin{array}{rr} \text{but} & x-y=10; \\ \hline \therefore \text{ by addition,} & 2x=22, \text{ or } -2, \\ \text{and} & x=11, \text{ or } -1; \\ \text{by subtraction,} & 2y=2, \text{ or } -22, \\ \text{and} & y=1, \text{ or } -11. \end{array}$$

$$\begin{array}{rr} \text{But if} & x-y=122, \\ \text{then} & x^2+y^2=10, \\ \text{and} & x^2-2xy+y^2=\overline{122}|^2; \\ \therefore \text{ by subtraction,} & 2xy=10-\overline{122}|^2; \\ \text{but} & x^2+y^2=10; \\ \hline \therefore \text{ by addition,} & x^2+2xy+y^2=20-\overline{122}|^2; \end{array}$$

and extracting the root, $x+y=\pm\sqrt{20-\overline{122}|^2}$;

but $x-y=122$;

$\therefore$ by addition, $2x=122\pm2\sqrt{5-\overline{61}|^2}$,

and $x=\ 61\pm\ \sqrt{-3716}$;

and by subtraction, $2y=-122\pm2\sqrt{-3716}$;

$\therefore\ y=-61\pm\sqrt{-3716}$.

18. Given $\left.\begin{array}{l} x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2 \\ \text{and } 8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14 \end{array}\right\}$, to find the values of x and y.

From the first equation, $x^{\frac{2}{3}}=2y^{\frac{1}{2}}$,

and, $\therefore\ \frac{1}{2}x^{\frac{2}{3}}=y^{\frac{1}{2}}$;

substituting this value in the second equation,

$$8x^{\frac{1}{3}}-\frac{1}{2}x^{\frac{2}{3}}=14,$$

and $16x^{\frac{1}{3}}-x^{\frac{2}{3}}=28$;

or (17. Cor. 1.) $x^{\frac{2}{3}}-16x^{\frac{1}{3}}=-28$;

completing the square, $x^{\frac{2}{3}}-16x^{\frac{1}{3}}+64=64-28=36$;

and extracting the root, $x^{\frac{1}{3}}-8=\pm6$;

$\therefore\ x^{\frac{1}{3}}=14$, or 2,

and $x=\overline{14}|^3$ or 8;

but $y^{\frac{1}{2}}=\frac{1}{2}x^{\frac{2}{3}}=98$, or 2;

$\therefore\ y=\overline{98}|^2$, or 4.

19. Given $\left.\begin{array}{l} x^{\frac{2}{3}}+y^{\frac{2}{3}}=3x \\ \text{and } x^{\frac{1}{3}}+y^{\frac{1}{3}}=\ x \end{array}\right\}$, to find the values of x and y.

Squaring the second equation, $\therefore\ x+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=x^2$;

but $x^{\frac{2}{3}}\qquad\quad +y^{\frac{2}{3}}=3x$;

$\therefore$ by subtraction, $x-x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}=x^2-3x$;

but from the second equation, $y^{\frac{1}{3}}=x-x^{\frac{1}{3}}$.

Let this value be substituted in the preceding equation,

$$\text{then } x - x^{\frac{3}{2}} + 2x^{\frac{3}{2}} - 2x = x^2 - 3x\,;$$

$$\text{and by transposition, } 2x = x^2 - x^{\frac{3}{2}}\,;$$

$$\text{and dividing by } x,\ 2 = x - x^{\frac{1}{2}}\,;$$

$$\text{completing the square, } x - x^{\frac{1}{2}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}} - \frac{1}{2} = \pm\frac{3}{2},$$

$$\text{and } x^{\frac{1}{2}} = 2, \text{ or } -1\,;$$

$$\therefore\ x = 4, \text{ or } 1,$$

$$\text{and } y^{\frac{1}{3}} = x - x^{\frac{1}{2}} = 2\,;$$

$$\therefore\ y = 8.$$

20. Given $\left.\begin{aligned} x + x^{\frac{1}{2}} &= \frac{y^2+y+2}{x^{\frac{1}{2}}} + 4 \\ \text{and } y + xy &= y^2 + 4y \end{aligned}\right\}$, to find the values of x and y.

From the first equation, $x^{\frac{3}{2}} + x - 4x^{\frac{1}{2}} = y^2 + y + 2$,

and from the second, $x = y + 3$.

Substituting this value for x in the former,

$$x^{\frac{3}{2}} + y + 3 - 4x^{\frac{1}{2}} = y^2 + y + 2,$$

$$\text{and by transposition, } x^{\frac{3}{2}} - 4x^{\frac{1}{2}} = y^2 - 1.$$

But since $x = y + 3$; $\therefore\ x - 4 = y - 1$,

by which equation let the preceding one be divided;

$$\therefore\ x^{\frac{1}{2}} = y + 1\,;$$

squaring both sides of this equation, $x = y^2 + 2y + 1$.

Equating therefore the two values of x,

$$y^2 + 2y + 1 = y + 3\,;$$

$$\therefore \text{ by transposition, } y^2 + y = 2\,;$$

$$\text{completing the square, } y^2 + y + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}\,;$$

extracting the root, $y+\frac{1}{2}=\pm\frac{3}{2}$;

$\therefore\ y=1$, or -2;

whence $x^{\frac{1}{2}}=y+1=2$, or -1,

and therefore $x=4$, or 1.

21. Given $\left.\begin{array}{l}\frac{x^2}{y^2}+\frac{y}{x}+\frac{x}{y}=6\frac{3}{4}-\frac{y^2}{x^2}\\ \text{and } x-y=2\end{array}\right\}$, to find the values of x and y.

By transposition, $\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{27}{4}$;

$\therefore$ adding 2 to each side, $\frac{x^2}{y^2}+2+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}=\frac{35}{4}$;

completing the square, $\overline{\frac{x}{y}+\frac{y}{x}}\Big|^2+\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{1}{4}=\frac{36}{4}$;

extracting the root, $\frac{x}{y}+\frac{y}{x}+\frac{1}{2}=\pm\frac{6}{2}$;

$\therefore\ \frac{x}{y}+\frac{y}{x}=\frac{5}{2}$, or $-\frac{7}{2}$;

$\therefore\ x^2+y^2=\frac{5xy}{2}$, or $-\frac{7xy}{2}$;

now from the second equation squared,

$x^2+y^2=2xy+4$;

$\therefore\ 2xy+4=\frac{5xy}{2}$, or $-\frac{7xy}{2}$;

whence by multiplication and transposition,

$xy=8$, or $-\frac{8}{11}$;

and since $x^2-2xy+y^2=4$,

and $\quad 4xy \quad =32$, or $-\frac{32}{11}$;

$\therefore$ by addition, $\quad x^2+2xy+y^2=36$, or $\frac{12}{11}$;

and extracting the root, $x+y=\pm 6$, or $\pm\dfrac{2\sqrt{3}}{\sqrt{11}}$;

$$\text{but } x-y=\quad 2;$$

$\therefore$ by addition, $\quad 2x=\quad 8$, or -4, or $2\pm\dfrac{2\sqrt{3}}{\sqrt{11}}$,

and $\quad x=4$, or -2, or $1\pm\dfrac{\sqrt{3}}{\sqrt{11}}$;

$\therefore$ by subtraction, $2y=4$, or -8, or $-2\pm\dfrac{2\sqrt{3}}{\sqrt{11}}$;

$$\therefore\ y=2,\text{ or } -4,\text{ or } -1\pm\frac{\sqrt{3}}{\sqrt{11}}.$$

22. Given $2x+y\quad =26-7\sqrt{2x+y+4}$ and $\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}=\dfrac{16}{15}+\dfrac{2x-\sqrt{y}}{2x+\sqrt{y}}$, to find the values of x and y.

Adding 4 to each side of the first equation,

$$2x+y+4=30-7\sqrt{2x+y+4};$$

$\therefore$ by transposition, $2x+y+4+7\sqrt{2x+y+4}=30$;
completing the square,

$$(2x+y+4)+7\sqrt{2x+y+4}+\frac{49}{4}=30+\frac{49}{4}=\frac{169}{4};$$

extracting the root, $\sqrt{2x+y+4}+\dfrac{7}{2}=\pm\dfrac{13}{2}$;

$$\therefore\ \sqrt{2x+y+4}=3,\text{ or } -10,$$

$$\text{and}\quad 2x+y+4=9,\text{ or } 100;$$

$$\therefore\ 2x+y\quad =5,\text{ or } 96.$$

Multiplying every term of the second equation by $\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}$, $\left|\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}\right|^2=\dfrac{16}{15}\cdot\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}+1$;

$\therefore$ by transposition, $\overline{\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}}\Big|^2 - \dfrac{16}{15}\cdot\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}=1;$

completing the square,

$$\overline{\frac{2x+\sqrt{y}}{2x-\sqrt{y}}}\Big|^2 - \frac{16}{15}\cdot\frac{2x+\sqrt{y}}{2x-\sqrt{y}}+\frac{64}{225}=1+\frac{64}{225}=\frac{289}{225};$$

extracting the root, $\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}-\dfrac{8}{15}=\pm\dfrac{17}{15};$

$$\therefore \frac{2x+\sqrt{y}}{2x-\sqrt{y}}=\frac{5}{3}, \text{ or } -\frac{3}{5}.$$

Let the former value be taken, then

$6x+3\sqrt{y}=10x-5\sqrt{y},$

and by transposition, $8\sqrt{y}=4x,$

and $2\sqrt{y}=x;$

but if the second value be taken, $\dfrac{2x+\sqrt{y}}{2x-\sqrt{y}}=-\dfrac{3}{5};$

$\therefore\ 10x+5\sqrt{y}=-6x+3\sqrt{y},$

and $16x=-2\sqrt{y};$

or $8x=-\sqrt{y}.$

Now $2x+y=5$, or 96; supposing the former,

and taking the first value of $2x=4\sqrt{y},$

$y+4\sqrt{y}=5;$

completing the square, $y+4\sqrt{y}+4=9;$

extracting the root, $\sqrt{y}+2=\pm 3;$

$\therefore\ \sqrt{y}=1,$ or $-5,$

and $y=1,$ or $25;$

but $x=2\sqrt{y}=2,$ or $-10.$

Again, taking the value, $2x = -\frac{1}{4}\sqrt{y}$,

$$y - \frac{1}{4}\sqrt{y} = 5;$$

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 5 + \frac{1}{64} = \frac{321}{64};$$

$\therefore$ extracting the root, $\sqrt{y} - \frac{1}{8} = \frac{\pm\sqrt{321}}{8}$,

$$\text{and } \sqrt{y} = \frac{1 \pm \sqrt{321}}{8};$$

$$\therefore y = \frac{161 \pm \sqrt{321}}{32},$$

$$\text{and } x = -\frac{1}{8}\sqrt{y} = \frac{-1 \mp \sqrt{321}}{64}.$$

Now taking the equation $2x + y = 96$, and the first value $2x = 4\sqrt{y}$;

then $y + 4\sqrt{y} = 96$;

completing the square, $y + 4\sqrt{y} + 4 = 100$,

and $\sqrt{y} + 2 = \pm 10$;

$\therefore \sqrt{y} = 8$, or -12;

and $\therefore y = 64$, or 144;

whence $x = 2\sqrt{y} = 16$, or -24.

Again, taking the value $2x = -\frac{1}{4}\sqrt{y}$;

then $y - \frac{1}{4}\sqrt{y} = 96$;

completing the square,

$$y - \frac{1}{4}\sqrt{y} + \frac{1}{64} = 96 + \frac{1}{64} = \frac{6145}{64};$$

extracting the root, $\sqrt{y} - \frac{1}{8} = \frac{\pm\sqrt{6145}}{8}$;

$$\text{and } \sqrt{y} = \frac{1 \pm \sqrt{6145}}{8};$$

$$\therefore\ y = \frac{3073 \pm \sqrt{6145}}{32},$$

$$\text{and } x = -\frac{1}{8}\sqrt{y} = \frac{-1 \mp \sqrt{6145}}{64}.$$

23. Given $\left.\begin{array}{l} \sqrt{y}+\sqrt{x} : \sqrt{y}-\sqrt{x} :: \sqrt{x}+2 : 1 \\ \text{and } \dfrac{\sqrt{y+2}}{\sqrt{x}} - 1 = \dfrac{3\sqrt{x}+1+\dfrac{\sqrt{y}}{\sqrt{x}}}{\sqrt{y}} \end{array}\right\}$,

to find the values of x and y.

From the first equation,

$$\sqrt{y} : \sqrt{x} :: \sqrt{x}+3 : \sqrt{x}+1;$$

$$\therefore\ \sqrt{xy}+\sqrt{y} = x+3\sqrt{x};$$

and from the second,

$$y+2\sqrt{y}-\sqrt{xy} = 3x+\sqrt{x}+\sqrt{y};$$

$\therefore$ by transposition, $y+\sqrt{y}-\sqrt{xy} = 3x+\sqrt{x}$;

but $\sqrt{y}+\sqrt{xy} = x+3\sqrt{x}$;

$\therefore$ by addition, $y+2\sqrt{y} = 4x+4\sqrt{x}$;

completing the square, $y+2\sqrt{y}+1 = 4x+4\sqrt{x}+1$,

and extracting the root, $\sqrt{y}+1 = \pm\,(2\sqrt{x}+1)$;

and $\therefore$ if the positive value be taken, $\sqrt{y} = 2\sqrt{x}$;

$$\therefore\ x+3\sqrt{x} = 2\sqrt{x}+2x,$$

$$\text{and } x = \sqrt{x};$$

$$\therefore\ \sqrt{x} = 1,$$

$$\text{and } x = 1;$$

$$\therefore \sqrt{y} = 2\sqrt{x} = 2,$$
$$\text{and } y = 4.$$

But if the negative value be taken,

$$\sqrt{y} + 1 = -2\sqrt{x} - 1;$$

by transposition, $\sqrt{y} = -2\sqrt{x} - 2$;

and if this value be substituted in the first equation,

$$-\sqrt{x} - 2 : -3\sqrt{x} - 2 :: \sqrt{x} + 2 : 1,$$
$$\text{or } \sqrt{x} + 2 : 3\sqrt{x} + 2 :: \sqrt{x} + 2 : 1;$$

and since the first term in this proportion is equal to the third, the second will be equal to the fourth;

$$\therefore 3\sqrt{x} + 2 = 1;$$

by transposition, $3\sqrt{x} = -1$;

$$\therefore \sqrt{x} = -\frac{1}{3},$$
$$\text{and } x = \frac{1}{9};$$

whence $\sqrt{y} = -2.(\sqrt{x} + 1) = -\frac{4}{3}$,

$$\text{and } y = \frac{16}{9}.$$

24. Given $\left.\begin{array}{l} y + \sqrt{\frac{y}{x}} = \frac{42}{x} \\ \text{and } \frac{x^2}{3} + \frac{x}{2\sqrt{y}} = \frac{54}{y} \end{array}\right\}$, to find the values of x and y.

Completing the square in the first equation,

$$y + \sqrt{\frac{y}{x}} + \frac{1}{4x} = \frac{42}{x} + \frac{1}{4x} = \frac{169}{4x},$$

and extracting the root, $\sqrt{y} + \frac{1}{2\sqrt{x}} = \frac{\pm 13}{2\sqrt{x}}$;

$$\therefore \sqrt{y} = \frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$$

$$\therefore y = \frac{36}{x}, \text{ or } \frac{49}{x}.$$

Again, from the second equation,

$$x^2 + \frac{3x}{2\sqrt{y}} = \frac{162}{y};$$

completing the square,

$$x^2 + \frac{3x}{2\sqrt{y}} + \frac{9}{16y} = \frac{162}{y} + \frac{9}{16y} = \frac{2601}{16y};$$

and extracting the root, $x + \frac{3}{4\sqrt{y}} = \pm\frac{51}{4\sqrt{y}};$

$$\therefore x = \frac{12}{\sqrt{y}}, \text{ or } \frac{-27}{2\sqrt{y}}.$$

$$\text{But } \sqrt{y} = \frac{6}{\sqrt{x}}, \text{ or } \frac{-7}{\sqrt{x}};$$

$$\text{whence, } x = \frac{12}{\sqrt{y}}, \text{ or } \frac{-27}{2\sqrt{y}} = 2\sqrt{x},$$

$$\text{or } \frac{-12\sqrt{x}}{7}, \text{ or } \frac{-9\sqrt{x}}{4}, \text{ or } \frac{27\sqrt{x}}{14};$$

$$\therefore \sqrt{x} = 2, \text{ or } \frac{-12}{7}, \text{ or } \frac{-9}{4}, \text{ or } \frac{27}{14}$$

$$\therefore x = 4, \text{ or } \frac{144}{49}, \text{ or } \frac{81}{16}, \text{ or } \frac{729}{196},$$

$$\text{and } y = \frac{36}{x}, \text{ or } \frac{49}{x} = 9, \text{ or } \frac{49}{4}, \text{ or } \frac{784}{81}, \text{ or } \frac{49 \times 196}{729}.$$

25. Given $\left.\begin{array}{l} \frac{x^2}{y^2} + \frac{2x+y}{\sqrt{y}} = 20 - \frac{y^2+x}{y} \\ \text{and } x + 8 = 4y \end{array}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$\frac{x^2}{y^2} + \frac{2x}{\sqrt{y}} + y + \frac{x}{y} + \sqrt{y} = 20;$$

completing the square,

$$\overline{\frac{x}{y} + \sqrt{y}}\Big|^2 + \left(\frac{x}{y} + \sqrt{y}\right) + \frac{1}{4} = 20 + \frac{1}{4} = \frac{81}{4};$$

extracting the root, $\frac{x}{y} + \sqrt{y} + \frac{1}{2} = \pm\frac{9}{2}$;

$$\therefore\ \frac{x}{y} + \sqrt{y} = 4, \text{ or } -5,$$

$$\text{and } x + y^{\frac{3}{2}} = 4y, \text{ or } -5y.$$

Let the former be taken.

Now from the second equation, $x + 8 = 4y$;

$\therefore$ by subtraction, $y^{\frac{3}{2}} - 8 = 0$,

or $y^{\frac{3}{2}} = 8$;

$\therefore\ y = 4$;

$\therefore\ x = 4y - 8 = 16 - 8 = 8$.

But if $x + y^{\frac{3}{2}} = -5y$,

and $x + 8 = 4y$;

$\therefore$ by subtraction, $8 - y^{\frac{3}{2}} = 9y$;

$\therefore$ by transposition, $8 - 8y = y + y^{\frac{3}{2}}$,

or $8.(1 - y) = y.(1 + y^{\frac{1}{2}}.)$

Dividing by $(1 + y^{\frac{1}{2}})$, $\therefore\ 8(1 - y^{\frac{1}{2}}) = y$;

$\therefore$ by transposition, $y + 8y^{\frac{1}{2}} = 8$;

completing the square, $y + 8y^{\frac{1}{2}} + 16 = 24$,

and extracting the root, $y^{\frac{1}{2}} + 4 = \pm 2\sqrt{6}$;

$\therefore\ y^{\frac{1}{2}} = -4 \pm 2\sqrt{6}$,

$$\text{and } y = 40 \mp 16\sqrt{6};$$

$$\therefore x = 4y - 8 = 152 \mp 64\sqrt{6}.$$

26. Given $\left.\begin{array}{l} 8x + 23y = 2x^3 + 2y^3 \\ \text{and } 34y + 6x^2 - 5y^2 = 13xy + 24 \end{array}\right\}$, to find the values of x and y.

From the second equation,

$$6x^2 - 13xy = 5y^2 - 34y + 24,$$

$$\text{and } x^2 - \frac{13}{6}.xy = \frac{5}{6}y^2 - \frac{34}{6}y + 4;$$

completing the square,

$$x^2 - \frac{13}{6}xy + \frac{169}{144}y^2 = \frac{169}{144}y^2 + \frac{5}{6}y^2 - \frac{34}{6}y + 4$$

$$= \frac{289}{144}.y^2 - \frac{34}{6}.y + 4;$$

extracting the root, $x - \frac{13}{12}y = \pm\left(\frac{17}{12}.y - 2\right)$;

and first, taking the positive value, $x = \frac{5}{2}y - 2$.

Let this value of x be substituted in the first equation,

$$\therefore 20y - 16 + 23y = \frac{125y^3}{4} - 75y^2 + 60y - 16 + 2y^3;$$

$$\therefore \text{ by transposition, } \frac{133y^3}{4} - 75y^2 + 17y = 0,$$

and dividing by $\frac{133y}{4}$, $y^2 - \frac{300}{133}y + \frac{68}{133} = 0$;

by transposition, $y^2 - \frac{300}{133}y = -\frac{68}{133}$;

completing the square,

$$y^2 - \frac{300}{133}.y + \overline{\frac{150}{133}}\Big|^2 = \frac{22500}{133|^2} - \frac{68}{133} = \frac{13456}{133|^2};$$

extracting the root, $y-\frac{150}{133}=\pm\frac{116}{133}$;

$$\therefore\ y=2,\ \text{or}\ \frac{34}{133};$$

$$\therefore\ x=\frac{5y}{2}-2=3,\ \text{or}\ \frac{-171}{133}.$$

But if the negative value be taken,

$$x=2-\frac{1}{3}y.$$

Let this value be substituted in the first equation,

$$\therefore\ 16-\frac{8}{3}y+23y=16-8y+\frac{4}{3}y^2-\frac{2y^3}{27}+2y^3;$$

$\therefore$ by transposition, $\frac{52y^3}{27}+\frac{4}{3}y^2=\frac{85}{3}y$,

and dividing by $\frac{52y}{27}$, $y^2+\frac{9}{13}y=\frac{765}{52}$;

completing the square,

$$y^2+\frac{9}{13}y+\left.\overline{\frac{9}{26}}\right|^2=\left.\overline{\frac{9}{26}}\right|^2+\frac{765}{52}=\frac{10026}{\left.\overline{26}\right|^2};$$

extracting the root, $y+\frac{9}{26}=\frac{\pm\sqrt{10026}}{26}$;

$$\therefore\ y=\frac{-9\pm\sqrt{10026}}{26}=\frac{-9\pm3\sqrt{1114}}{26},$$

$$\text{and}\ x=2-\frac{1}{3}y=\frac{55\mp\sqrt{1114}}{26}.$$

27. Given $\left.\begin{array}{r}\frac{x}{y}-8\sqrt{x^2-9xy^2}=9y-16xy\\ \text{and}\ 5x=4+25y^2\end{array}\right\}$, to find the values of x and y.

From the first equation,

$$x-8x^{\frac{1}{2}}y\sqrt{x-9y^2}=9y^2-16xy^2;$$

∴ by transposition,

$$(x-9y^2) - 8x^{\frac{1}{2}}y\sqrt{x-9y^2}+16xy^2=0;$$

extracting the root, $\sqrt{x-9y^2}-4x^{\frac{1}{2}}y=0,$

and $\sqrt{x-9y^2}=4x^{\frac{1}{2}}y,$

or $x-9y^2=16xy^2;$

and $x=(9+16x).y^2;$

$$\therefore\ y^2=\frac{x}{9+16x}.$$

But from the second equation, $y^2=\dfrac{5x-4}{25};$

$$\therefore\ \frac{5x-4}{25}=\frac{x}{9+16x},$$

and $80x^2-19x-36=25x;$

∴ by transposition, $80x^2-44x=36,$

and $x^2-\dfrac{22}{40}.x=\dfrac{18}{40};$

completing the square,

$$\overline{x^2-\frac{22}{40}.x+\frac{11}{40}}\Big|^2=\frac{121}{1600}+\frac{18}{40}=\frac{841}{1600};$$

extracting the root, $x-\dfrac{11}{40}=\pm\dfrac{29}{40};$

$$\therefore\ x=1,\ \text{or}\ -\frac{9}{20};$$

$$\therefore\ y^2=\frac{5x-4}{25}=\frac{1}{25},\ \text{or}\ -\frac{1}{4};$$

$$\therefore\ y=\pm\frac{1}{5},\ \text{or}\ \pm\frac{1}{2}\sqrt{-1}.$$

28. Given $\left.\begin{array}{l}16x-y^{\frac{1}{2}}=6y^{\frac{1}{4}}x^{\frac{1}{2}}\\ \text{and } \dfrac{x^4}{y}-\dfrac{12}{x^2}=\dfrac{x}{\sqrt{y}}\end{array}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$16x = y^{\frac{1}{2}} + 6y^{\frac{1}{4}}x^{\frac{1}{2}};$$

completing the square, $25x = y^{\frac{1}{2}} + 6y^{\frac{1}{4}}x^{\frac{1}{2}} + 9x$;

extracting the root, $\pm 5x^{\frac{1}{2}} = y^{\frac{1}{4}} + 3x^{\frac{1}{2}}$;

$$\therefore \quad 2x^{\frac{1}{2}}, \text{ or } -8x^{\frac{1}{2}} = y^{\frac{1}{4}},$$

and therefore $4x$, or $64x = y^{\frac{1}{2}}$.

Now from the second equation, $\frac{x^4}{y} - \frac{x}{\sqrt{y}} = \frac{12}{x^2}$;

completing the square, $\frac{x^4}{y} - \frac{x}{\sqrt{y}} + \frac{1}{4x^2} = \frac{12}{x^2} + \frac{1}{4x^2} = \frac{49}{4x^2}$;

extracting the root, $\frac{x^2}{\sqrt{y}} - \frac{1}{2x} = \pm \frac{7}{2x}$;

$$\therefore \frac{x^2}{\sqrt{y}} = \frac{4}{x}, \text{ or } -\frac{3}{x},$$

and $\frac{x^3}{\sqrt{y}} = 4$, or -3.

Now $\sqrt{y} = 4x$; or $64x$;

which, substituted in the last equation, gives,

$$\frac{x^2}{4}, \text{ or } \frac{x^2}{64} = 4, \text{ or } -3;$$

$$\therefore \frac{x}{2}, \text{ or } \frac{x}{8} = \pm 2, \text{ or } \pm\sqrt{-3};$$

whence $x = \pm 4$, or ± 16, or $\pm 2\sqrt{-3}$, or $\pm 8\sqrt{-3}$,

and $y = 256$, or $\overline{256}|^2$, or -192, or $-3 \times \overline{64}|^4$.

29. Given $\left.\begin{matrix} y^2 - 64 = 8x^{\frac{1}{2}}y \\ \text{and } y - 4 = 2y^{\frac{1}{2}}x^{\frac{1}{2}} \end{matrix}\right\}$, to find the values of x and y.

From the first equation, by transposition,

$$y^2 - 8x^{\frac{1}{2}}y = 64;$$

completing the square, $y^2 - 8x^{\frac{1}{2}}y + 16x = 16x + 64$;

extracting the root, $y - 4x^{\frac{1}{2}} = \pm 4\sqrt{x+4}$,

and $y = 4x^{\frac{1}{2}} \pm 4\sqrt{x+4}$.

Also from the second equation, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} = 4$;

completing the square, $y - 2y^{\frac{1}{2}}x^{\frac{1}{2}} + x = x + 4$;

extracting the root, $y^{\frac{1}{2}} - x^{\frac{1}{2}} = \pm\sqrt{x+4}$;

$\therefore 4y^{\frac{1}{2}} = 4x^{\frac{1}{2}} \pm 4\sqrt{x+4} = y$, from the last equation;

$\therefore 4 = y^{\frac{1}{2}}$, and $16 = y$;

And from the second equation, $x^{\frac{1}{2}} = \dfrac{y-4}{2y^{\frac{1}{2}}} = \dfrac{12}{8} = \dfrac{2}{3}$;

$$\therefore x = \frac{9}{4}.$$

30. Given $\left.\begin{array}{r}\sqrt{5\sqrt{x}+5\sqrt{y}}+\sqrt{y}=10-\sqrt{x} \\ \text{and } \sqrt{x^5}+\sqrt{y^5}=275\end{array}\right\}$, to find the values of x and y.

From the first equation,

$$\sqrt{x}+\sqrt{y}+\sqrt{5}\,.\sqrt{\sqrt{x}+\sqrt{y}}=10;$$

completing the square,

$$\sqrt{x}+\sqrt{y}+\sqrt{5}\,.\sqrt{\sqrt{x}+\sqrt{y}}+\frac{5}{4}=10+\frac{5}{4}=\frac{45}{4};$$

extracting the root, $\sqrt{\sqrt{x}+\sqrt{y}}+\dfrac{\sqrt{5}}{2} = \pm\dfrac{3\sqrt{5}}{2}$,

and $\sqrt{\sqrt{x}+\sqrt{y}}=\sqrt{5}$, or $-2\sqrt{5}$;

$\therefore \sqrt{x}+\sqrt{y}=5$, or 20, supposing the former;

∴ by involution,

$$x^{\frac{5}{2}} + 5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 + y^{\frac{5}{2}} = 3125;$$

but $x^{\frac{5}{2}} \qquad\qquad + y^{\frac{5}{2}} = 275;$

∴ by subtraction,

$$5x^2y^{\frac{1}{2}} + 10x^{\frac{3}{2}}y + 10xy^{\frac{3}{2}} + 5x^{\frac{1}{2}}y^2 = 2850;$$

$$\text{or } 5x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}\right) = 2850;$$

$$\text{and } x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{3}{2}} + 2xy^{\frac{1}{2}} + 2x^{\frac{1}{2}}y + y^{\frac{3}{2}}\right) = 570;$$

$$\text{but } x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{3}{2}} + 3xy^{\frac{1}{2}} + 3x^{\frac{1}{2}}y + y^{\frac{3}{2}}\right) = 125x^{\frac{1}{2}}y^{\frac{1}{2}}.$$

∴ by subtraction, $x^{\frac{1}{2}}y^{\frac{1}{2}} \times \left(xy^{\frac{1}{2}} + x^{\frac{1}{2}}y\right) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$

$$\text{or } x^{\frac{1}{2}}y^{\frac{1}{2}} \times x^{\frac{1}{2}}y^{\frac{1}{2}} \times \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$$

$$\text{or } 5xy = 125x^{\frac{1}{2}}y^{\frac{1}{2}} - 570;$$

$$\therefore\ xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} = -114;$$

completing the square,

$$\overline{xy - 25x^{\frac{1}{2}}y^{\frac{1}{2}} + \frac{25}{2}}\Big|^2 = \frac{625}{4} - 114 = \frac{169}{4};$$

$$\text{extracting the root, } x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{25}{2} = \pm\frac{13}{2};$$

$$\therefore\ x^{\frac{1}{2}}y^{\frac{1}{2}} = 19, \text{ or } 6.$$

$$\text{but } x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 25,$$

$$\text{and } 4x^{\frac{1}{2}}y^{\frac{1}{2}} \qquad = 24, \text{ or } 76;$$

∴ by subtraction, $x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 1$, or -51.

extracting the root, $x^{\frac{1}{2}} - y^{\frac{1}{2}} = \pm 1$, or $\pm\sqrt{-51}$;

$$\text{but } x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5;$$

∴ by addition, $2x^{\frac{1}{2}} = 6$, or 4, or $5 \pm \sqrt{-51}$;

$$\therefore\ x^{\frac{1}{2}} = 3, \text{ or } 2, \text{ or } \frac{5 \pm \sqrt{-51}}{2},$$

$$\text{and } x = 9, \text{ or } 4, \text{ or } \frac{-13 \pm \sqrt{-51}}{2};$$

by subtraction, $2y^{\frac{1}{2}} = 4$, or 6, or $5 \mp \sqrt{-51}$;

$$\therefore\ y^{\frac{1}{2}} = 2, \text{ or } 3, \text{ or } \frac{5 \mp \sqrt{-51}}{2};$$

$$\therefore\ y = 4, \text{ or } 9, \text{ or } \frac{-13 \mp \sqrt{-51}}{2}; *$$

* The other case, where $\sqrt{x} + \sqrt{y} = 20$, is solved in the same manner.

Sect. VI.

On the Solution of Problems which involve Simple Equations.

(30.) The solution of a problem, or method of discovering by analysis quantities which will answer its several conditions, is performed by assuming algebraic symbols to represent the quantities sought, and by deducing equations from the application of these, in the same manner as if they were known quantities, to the conditions of the problem. The independent equations derived from this process, if the conditions be properly limited, will equal in number the unknown quantities assumed; and from the solution of these several equations by the rules already given (23. 27. 29), the values of the algebraic symbols will be determined. Whether these values are correct, may be determined synthetically, by applying them instead of their respective symbols to the several conditions of the problem.

If the conditions of the problem are not properly limited, that is, are not sufficient in number, or not sufficiently independent of each other, the resulting equations will either exceed in number the unknown quantities, and will therefore some of them be identical or inconsistent, or will be fewer in number than the unknown quantities, and consequently will admit of an indefinite number of answers.

In many cases, instead of assuming a symbol to represent each of the required quantities, it is convenient to assume one only, and from the conditions of the problem to deduce expressions for the others in terms of that one and known quantities. And as the number of conditions ought to be one more than the number of quantities thus expressed, there will remain one to be stated in an equation; from which the value of the unknown quantity may be determined (22. 27. 28): and this being substituted in the other expressions, their value also may be discovered.

Examples of the Solution of Problems producing simple Equations involving only one unknown quantity.

1. What number is that, to the double of which if 18 be added, the sum will be 82?

Let $x =$ the number required.

Then by the problem, $2x+18=82$

$\therefore$ by transposition, $2x=64$

and $x=32$

2. What number is that, to the double of which if 44 be added, the sum is equal to four times the required number?

Let $x =$ the number.

Then $2x+44=4x$, by supposition;

$\therefore$ by transposition $44=2x$

and $22=x$.

3. What number is that, the double of which exceeds its half by 6?

Let $x =$ the number.

then by the problem, $2x - \frac{x}{2} = 6$,

$$\therefore\ 4x - x = 12,$$
$$\text{or } 3x = 12,$$
$$\therefore\ x = 4.$$

4. From two towns which are 187 miles distant, two travellers set out at the same time with an intention of meeting. One of them goes 8 miles, and the other 9 miles a day. In how many days will they meet?

Let $x =$ the number of days required;
then $8x =$ the number of miles one travelled,
and $9x =$ the number the other travelled;

and since they meet, they must together have travelled the whole distance,

$$\text{consequently } 8x + 9x = 187,$$
$$\text{or } 17x = 187,$$
$$\therefore\ x = 11.$$

5. A Gentleman meeting 4 poor persons distributed five shillings amongst them: to the second he gave twice, to the third thrice, and to the fourth four times as much as to the first. What did he give to each?

Let $x =$ the pence he gave to the first,
$\therefore\ 2x =$ the pence given to the second,
and $3x =$ - - - to the third,
$4x =$ - - - to the fourth.

$$\therefore\ x + 2x + 3x + 4x = 60,$$
$$\text{or } 10x = 60,$$
$$\therefore\ x = 6,$$

and $\therefore$ he gave 6, 12, 18, 24 pence respectively to them.

$\therefore\ x+x+4+x+7+x+9=32,$

and by transposition, $4x=12,$

$\therefore\ x=3.$

$\therefore$ they received 12, 10, 7, and 3 shillings respectively.

12. A Father taking his 4 sons to school divided a certain sum amongst them. Now the third had 9 shillings more than the youngest; the second 12 shillings more than the third; and the eldest 18 shillings more than the second; and the whole sum was 6 shillings more than 7 times the sum which the youngest received. How much had each?

Suppose the youngest received x shillings.

then the third received $x+9$ - - -

the second - - $x+21$ - - -

and the eldest - - $x+39$ - - -

$\therefore\ x+x+9+x+21+x+39=7x+6,$

$\therefore$ by transposition, $63=3x,$

and $\therefore\ 21=x.$

consequently they received 21, 30, 42, and 60 shillings respectively.

13. A sum of money was to be divided amongst six poor persons; the second received 10*d.* the third 14*d.* the fourth 25*d.* the fifth 28*d.* and the sixth 33*d.* less than the first. Now the sum distributed was 10*d*, more than the treble of what the first received. What did each receive?

Let $x=$ what the first received,

$\therefore\ x-10=$ - - - second - -

$x-14=$ - - - third - - -

$x-25=$ - - - fourth - -

$x-28=$ - - - fifth - - -

$x-33=$ - - - sixth - - -

The sum of which $=6x-110=3x+10$ by supposition.

$\therefore$ by transposition, $3x = 120$,

and $x = 40$.

$\therefore$ they received 40, 30, 26, 15, 12, 7 pence respectively.

---

14. It is required to divide the number 99 into five such parts, that the first may exceed the second by 3; be less than the third by 10; greater than the fourth by 9; and less than the fifth by 16.

Let $x =$ the first part,

$\therefore x - 3 =$ second,

$x + 10 =$ third,

$x - 9 =$ fourth,

$x + 16 =$ fifth.

$\therefore x + x - 3 + x + 10 + x - 9 + x + 16 = 99$,

or $5x + 14 = 99$,

$\therefore$ by transposition, $5x = 85$,

and $x = 17$.

$\therefore$ the parts are 17, 14, 27, 8, and 33.

---

15. What two numbers are those whose sum is 59, and difference 17?

Let $x =$ the less,

$\therefore x + 17 =$ the greater.

and $\therefore x + x + 17 = 59$,

by transposition, $2x = 42$,

and $x = 21$ the less,

$\therefore$ the greater $= 38$.

---

16. What number is that, the treble of which increased by 12, shall as much exceed 54 as that treble is below 144?

Let $x =$ the number.

$\therefore 3x + 12 - 54 = 144 - 3x$ by supposition.

$\therefore$ by transposition, $6x = 186$,

and $x = 31$.

17. Two persons began to play with equal sums of money: the first lost 14 shillings, the other won 24 shillings; and then the second had twice as many shillings as the first. What sum had each at first?

Let $x=$ the sum;

Then $\left.\begin{array}{l} x-14 \\ \text{and } x+24 \end{array}\right\} =$ the sums each had after playing;

$\therefore$ by problem $2x-28=x+24$

$\therefore\ x=52.$

18. At a certain election 943 men voted, and the candidate chosen had a majority of 65. How many voted for each?

Let $x=$ the number of votes the unsuccessful candidate had;

$\therefore\ x+65=$ the number the successful one had.

$\therefore\ x+x+65=943$;

by transposition, $2x=878$,

and $x=439$.

$\therefore$ the numbers were 439 and 504.

19. Two Robbers after plundering a house found that they had 35 guineas between them; and that if one of them had had 4 guineas more, he should have had twice as many as the other. How many had each?

Let $x=$ the number one had,

$\therefore\ 35-x=$ the number the other had,

and $35-x+4=2x$.

by transposition, $39=3x$,

and $13=x$.

$\therefore$ they had 13 and 22 guineas respectively.

20. A Mercer having cut 19 yards from each of three equal pieces of silk, and 17 from another of the same length, found that the remnants taken together were 142 yards. What was the length of each piece?

Let $x=$ the length,

$\therefore x-19=$ the length of each of the 3 remnants,

and $x-17=$ the length of the other.

then $3\,.\,(x-19)+x-17=142$,

or $3x-57+x-17=142$.

by transposition, $4x=216$,

$\therefore x=54$.

21. A Farmer has two flocks of sheep, each containing the same number. From one of these he sells 39, and from the other 93; and finds just twice as many remaining in one as in the other. How many did each flock originally contain?

Let $x=$ the number required.

Then $x-39$ and $x-93$ are the numbers remaining;

$\therefore x-39=2x-186$;

and by transposition, $147=x$.

22. Bought 12 yards of cloth for £.10. 14*s*. For part of it I gave 19 shillings a yard, and for the rest 17 shillings a yard. How many yards of each were bought?

Let $x=$ the number of yards at 19*s*. per yard;

$\therefore 12-x=$ the number at 17*s*.

and $19x=$ the price of the former,

and $17\,.\,(12-x)=$ the price of the latter.

$\therefore 19x+204-17x=214$.

and by transposition, $2x=10$,

and, $x=5$.

$\therefore$ there were 5 yards at 19 shillings, and 7 at 17 shillings.

23. Divide the number 197 into two such parts that four times the greater may exceed five times the less by 50.

Let $x=$ the less,
and $\therefore$ $197-x=$ the greater.
Then $788-4x=5x+50$;
and by transposition, $738=9x$.
and $82=x$;
$\therefore$ the greater $=115$.

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24. A Courier, who travels 60 miles a day, had been dispatched five days, when a second was sent to overtake him; in order to which, he must go 75 miles a day: In what time will he overtake the former.

Let $x=$ the number of days the second courier
then $x+5=$ the number the first travels; [travels.
$\therefore$ $75x=$ the number of miles the second travels,
and $60(x+5)=$ the number the first travels.

But by the supposition they both travelled the same number of miles;

$\therefore$ $75x=60x+300$;
by transposition, $15x=300$,
and $x=20$.

---

25. After *A* had lost 10 guineas to *B*, he wanted only 8 guineas in order to have as much money as *B*: and together they had 60 guineas. What money had each at first?

Let $x=$ the number of guineas *A* had;
$\therefore$ $60-x=$ the number - - - *B* had.
Then after playing *A* had $x-10$, and *B* had $70-x$;
$\therefore$ $x-10+8=70-x$.
by transposition, $2x=72$,
and $x=36$.
$\therefore$ they had 36 and 24 guineas respectively.

26. A and B began trade with equal stocks. In the first year A tripled his stock and had £.27 to spare; B doubled his stock and had £.153 to spare. Now the amount of both their gains was five times the stock of either. What was that stock?

$$\text{Let } x = \text{ the stock;}$$
$$\text{then } 3x + 27 = A\text{'s stock at the end of the year,}$$
$$\therefore\ 2x + 27 = \text{his gain.}$$
$$\text{and } 2x + 153 = B\text{'s stock at the end of the year;}$$
$$\therefore\ x + 153 = B\text{'s gain.}$$
$$\therefore\ 5x = 2x + 27 + x + 153.$$
$$\text{by transposition } 2x = 180,$$
$$\text{and } x = 90.$$

27. Two workmen A and B were employed together for 50 days, at 5 shillings per day each. A spent sixpence a day less than B did, and at the end of the fifty days he found he had saved twice as much as B, and the expence of two days over. What did each spend per day?

$$\text{Let } x = \text{ what } A \text{ spent per day (in pence);}$$
$$\therefore\ 60 - x = \text{ what he saved per day,}$$
$$\text{and } 54 - x = \text{ what } B \text{ saved per day.}$$
$$\text{and } \therefore\ 3000 - 50x = 5400 - 100x + 2x.$$
$$\text{by transposition, } 48x = 2400,$$
$$\text{and } x = 50;$$
$$\therefore\ A \text{ spent 50 pence, and } B \text{ 56 pence a day.}$$

28. A and B began to trade with equal sums of money. In the first year A gained 40 pounds and B lost 40; but in the second A lost one-third of what he then had, and B gained a sum less by 40 pounds than twice the sum that A had lost; when it appeared that B had twice as much money as A. What money did each begin with?

$$\therefore (2x+x=)\ 3x=210,$$
$$\text{and } x=70;$$

$\therefore$ they had 140 and 70 pounds respectively.

35. A prize of £.2329 was divided between two persons *A* and *B*, whose shares therein were in proportion of 5 to 12. What was the share of each?

Let $5x=$ *A*'s share,
then $12x=$ *B*'s share;
$$\therefore (5x+12x=)\ 17x=2329,$$
$$\text{and } x=137;$$

$\therefore$ their shares were 685 and 1644 pounds respectively.

36. A sum of money is to be shared between two persons *A* and *B*, so that as often as *A* receives 9 pounds, *B* takes 4. Now it happens that *A* receives 15 pounds more than *B*. What are their respective shares?

Since for every £.9 that *A* receives, *B* receives £.4,
Let $9x=$ the whole sum *A* receives;
$\therefore$ $4x=$ the whole sum *B* receives.
$$\therefore 9x=4x+15;$$
and by transposition, $5x=15$,
$$\therefore x=3;$$

$\therefore$ *A* receives £.27, and *B*, £.12

37. A Gentleman gave to 3 persons 98 pounds. The second received five-eighths of the sum given to the first, and the third one-fifth of what the second had. What did each receive?

Let $8x=$ the number of pounds the first received;
$\therefore$ $5x=$ - - - - - second - -
and $x=$ - - - - - third - -

$\therefore\ (8x+5x+x=)\ 14x=98,$

and $x=7;$

$\therefore$ they received 56, 35, and 7 pounds, respectively.

38. A person bought two casks of beer, one of which held exactly three times as much as the other. From each of these he drew four gallons, and then found that there were four times as many gallons remaining in the larger, as in the other. How many were there in each at first?

Let $3x=$ the number of gallons in the larger;

and $\therefore\ x=$ the number in the smaller;

$\therefore\ 4.(x-4)=3x-4;$

by transposition, $x=12;$

$\therefore$ they held 36 and 12 gallons, respectively.

39. A man at a party at cards betted three shillings to two upon every deal. After twenty deals he won five shillings. How many deals did he win?

Let $x=$ the number of deals he won;

$\therefore\ 20-x=$ the number he lost;

$\therefore\ 2x=$ the money won,

and $3.(20-x)=$ the money lost;

whence $2x-3.(20-x)=5;$

$\therefore$ by transposition, $5x=65,$

and $x=13.$

40. What two numbers are as 2 to 3; to each of which if 4 be added, the sums will be as 5 to 7?

Let $2x$ and $3x$ be the numbers;

$\therefore\ 2x+4:3x+4::5:7,$

and (21) $14x+28=15x+20;$

by transposition, $8=x;$

and $\therefore$ the numbers are 16 and 24.

41. A sum of money was divided between two persons *A* and *B*, so that the share of *A* was to that of *B* as 5 to 3; and exceeded five-ninths of the whole sum by 50 pounds. What was the share of each person?

Let $5x =$ *A*'s share;
$\therefore 3x =$ *B*'s share,
and $8x =$ the whole sum;
$\therefore 5x = \frac{5}{9} . 8x + 50$,
or $x = \frac{8}{9} . x + 10$,
and $9x = 8x + 90$;
$\therefore$ by transposition, $x = 90$,
and the sums were 450 and 270 pounds.

42. Being sent to market to buy a certain quantity of meat, I found that if I bought beef, which was then 4 pence a pound, I should lay out all the money I was entrusted with; but if I bought mutton which was then threepence halfpenny a pound, I should have two shillings left. How much meat was sent for?

Let $2x =$ the number of pounds;
$\therefore 8x =$ the price of $2x$ lbs. of beef,
and $7x =$ the price of $2x$ lbs. of mutton,
and $8x = 7x + 24$;
$\therefore x = 24$;
whence 48 lbs. were sent for.

43. A fish was caught, whose tail weighed 9 lbs.; his head weighed as much as his tail, and half his body; and his body weighed as much as his head and tail. What did the fish weigh?

Let $2x =$ the number of lbs. the body weighed;
then $9 + x =$ the weight of the tail;

$\therefore 9+9+x=2x$;

by transposition, $18=x$;

$\therefore$ the fish weighed $36+27+9=72$ lbs.

44. The joint stock of two partners whose particular shares differed by 40 pounds was to the share of the lesser as 14 to 5. Required the shares.

Suppose $14x=$ the joint stock;

$\therefore 5x=$ the less,

and $9x=$ the greater;

$\therefore 9x=5x+40$;

by transposition, $4x=40$,

and $x=10$;

$\therefore$ the shares are 90 and 50 pounds, respectively.

45. A Bankrupt owed to two Creditors 140 pounds; the difference of the debts was to the greater as 4 to 9. What were the debts?

Let $4x=$ the difference of the debts;

and $\therefore 9x=$ the greater,

and $5x=$ the less;

$\therefore (9x+5x=)\ 14x=140$,

and $x=10$;

$\therefore$ the debts are 90 and 50 pounds.

46. A Gentleman employed two labourers at different times, one for 3 shillings, and the other for 5 shillings a day. Now the number of days added together was 40; and they each received the same sum. How many days was each employed?

Let $x=$ the number of days the second was employed;

$\therefore\ 40-x=$ the number the first was employed;

$\therefore\ 5x=$ the sum received by the second,

and $3\,.\,(40-x)=$ the sum received by the first;

$$\therefore\ 5x=3\,.\,(40-x);$$

by transposition, $8x=120$,

and $x=15$;

$\therefore$ the second was employed 15, and the first 25 days.

47. Some persons agreed to give sixpence each to a waterman for carrying them from London to Gravesend; but with this condition, that for every other person taken in by the way, three pence should be abated in their joint fare. Now the waterman took in three more than a fourth part of the number of the first passengers, in consideration of which he took of them but five pence each. How many persons were there at first?

Let $4x=$ the number of passengers at first;

then $x+3=$ the number taken in,

and $3x+9=$ the sum deducted from their joint fare;

$$\therefore\ 24x-(3x+9)=20x;$$

by transposition, $x=9$;

consequently there were 36 passengers.

48. In a mixture of wine and cyder, half of the whole + 25 gallons was wine, and one-third of the whole − 5 gallons was cyder. How many gallons were there of each?

Let $6x=$ the number of gallons in all;

$\therefore\ 3x+25=$ the number of gallons of wine,

and $2x-5=$ the number of gallons of cyder;

$$\therefore\ 6x=3x+25+2x-5;$$

by transposition, $x=20$;

consequently there were 85 gallons of wine, and 35 of cyder.

49. *A* and *B* engaged in trade, *A* with £.240, and *B* with £.96. *A* lost twice as much as *B*; and upon settling their accounts it appeared that *A* had three times as much remaining as *B*. How much did each lose?

Let $x =$ what *B* lost;
$\therefore 96 - x =$ what he had remaining;
then $2x =$ what *A* lost,
and $240 - 2x =$ what he had remaining;
$\therefore 240 - 2x = 3 \cdot (96 - x)$
by transposition, $x = 48$;
$\therefore$ *A* lost £.96, and *B* lost £.48.

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50. Four places are situated in the order of the four letters *A*, *B*, *C*, *D*. The distance from *A* to *D* is 34 miles, the distance from *A* to *B* : distance from *C* to *D* :: 2 : 3, and one-fourth of the distance from *A* to *B* added to half the distance from *C* to *D* is three times the distance from *B* to *C*. What are the respective distances?

Let $2x =$ the distance from *A* to *B*;
$\therefore 3x =$ the distance from *C* to *D*.

$$\text{and } \left(\frac{x}{2} + \frac{3x}{2} =\right) 2x = 3 \cdot BC;$$

$$\therefore BC = \frac{2}{3}x,$$

$$\text{and } \left(2x + 3x + \frac{2}{3} \cdot x =\right) \frac{17x}{3} = 34;$$

$$\therefore \frac{x}{3} = 2, \text{ and } x = 6;$$

whence $AB = 12$, $BC = 4$, and $CD = 18$.

51 A Field of wheat and oats which contained 20 acres was put out to a labourer to reap for six guineas, the wheat at 7 shillings an acre, and the oats at 5 shillings. Now the labourer falling ill, reaped only the wheat. How much money ought he to receive according to the bargain?

Let $x=$ the number of acres of wheat;
then $20-x=$ the number of acres of oats;
and $7x=$ the price of reaping the wheat (in shill^s.)
and $100-5x=$ the price of reaping the oats;

$$\therefore\ 7x+100-5x=126;$$

by transposition $2x=26$,

and $x=13$;

$\therefore$ he ought to receive £4. 11*s*.

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52. A General having lost a battle found that he had only half his army + 3600 men left, fit for action; one-eighth of his men + 600 being wounded, and the rest, which were one-fifth of the whole army, either slain, taken prisoners, or missing. Of how many men did his army consist?

Let $x=$ the number required;

$\therefore\ \frac{x}{2}+3600=$ the number fit for service;

$\frac{x}{8}+600=$ the number wounded,

and $\frac{x}{5}=$ the number missing;

$$\therefore\ x=\frac{x}{2}+3600+\frac{x}{8}+600+\frac{x}{5};$$

by transposition, $\frac{3x}{8}-\frac{x}{5}=4200$,

and $(15x-8x=)\ 7x=168000$,

and $x=24000$.

53. Three men, A, B, and C entered into partnership; A paid in as much as B and one-third of C; B paid in as much as C and one-third of A; and C paid in £.10 and one-third of A. What did each man contribute to the stock?

$$\text{Let } 3x = \text{ the sum } A \text{ contributed};$$
$$\therefore\ 10+x = \text{ - - } C \text{ - - - -}$$
$$\text{and } 10+2x = \text{ - - } B \text{ - - - -}$$
$$\therefore\ 3x = 10+2x+\frac{10+x}{3};$$
$$\text{by transposition, } \frac{2x}{3} = 10+\frac{10}{3},$$
$$\text{and } 2x = 40;$$
$$\therefore\ x = 20.$$

and the sums contributed were £.60, £.50, and £.30, by A, B, C, respectively.

---

54. It is required to divide the number 91 into two such parts that the greater being divided by their difference, the quotient may be 7.

$$\text{Let } x = \text{ the greater};$$
$$\therefore\ 91-x = \text{ the less,}$$
$$\text{and } \frac{x}{2x-91} = 7;$$
$$\therefore\ x = 14x - 637;$$
$$\text{by transposition, } 637 = 13x;$$
$$\text{and } \therefore\ 49 = x;$$
$$\therefore \text{ the parts are 49 and 42.}$$

---

55. From each of 16 coins an artist filed the worth of half a crown, and then offered them in payment for their original value: but being detected, the pieces were found to be really worth no more than 8 guineas. What was their original value?

Let $x=$ the number of sixpences each was worth;

$\therefore\ x-5=$ the number each was worth after filing;

$\therefore\ 16.(x-5)=336.$

by transposition, $16x=416$,

and $x=26=13$ shillings.

56. *A* and *B* made a joint stock of £.833, which, after a successful speculation, produced a clear gain of £.153. Of this *B* had £.45 more than *A*. What did each person contribute to the stock?

Let $x=$ the sum brought in by *B*;

then $833 : x :: 153 :$ *B*'s gain $=\frac{9x}{49}$

$\therefore$ *A*'s gain $=\frac{9x}{49}-45$,

and $\frac{9x}{49}+\frac{9x}{49}-45=153$

by transposition, $\frac{18x}{49}=198$;

$\therefore\ x=\frac{49\times198}{18}=49\times11=539$,

whence, *B* brought in £.539 and *A* £.294.

57. Sold a quantity of tobacco for 19 shillings, part at 1 shilling a pound, and the rest at 15 pence. Now the first part was to the latter :: $\frac{3}{4}$: $\frac{2}{3}$ How much was sold of each?

Since $\frac{3}{4} : \frac{2}{3} :: 9 : 8$,

Let $9x=$ the number of lbs. of the former;

$\therefore\ 8x=$ the number of lbs. of the latter;

$\therefore\ 9x=$ the number of shillings the first sold for,

and $8x\times\frac{5}{4}=10x=$ the number of shillings the second sold for.

$$\therefore\ (10x+9x=)\ 19x=19,$$
$$\text{and}\ x=1;$$

∴ there were at 9 lbs. at 1 shilling, and 8 lbs. at 15 pence.

58. A Gentleman gave in charity £.46; a part thereof in equal portions to 5 poor men, and the rest in equal portions to 7 poor women. Now a man and a woman had between them £.8. What was given to the men, and what to the women?

Let $5x=$ the number of pounds the men received;
$\therefore\ 46-5x=$ the number the women received;
$\therefore\ x=$ the sum one man received,
and $8-x=$ the sum one woman received;

$$\therefore\ 56-7x=46-5x;$$
by transposition, $2x=10$,
and $x=5$;

∴ the men received £.25, and the women £.21.

59. Suppose that for every 10 sheep a farmer kept, he should plough an acre of land, and be allowed one acre of pasture for every 4 sheep. How many sheep may that person keep who farms 700 acres?

Let $x=$ the number of sheep required;

then $10 : x :: 1 :$ the number of acres ploughed $=\frac{x}{10}$,

and $4 : x :: 1 :$ the number of acres of pasture $=\frac{x}{4}$;

$$\therefore\ \frac{x}{10}+\frac{x}{4}=700,$$

and $(2x+5x=)\ 7x=20\times700$;
$$\therefore\ x=20\times100=2000.$$

60. A person being asked the hour, answered that it was between five and six; and the hour and minute-hands were together. What was the time?

Let $x=$ the time past 5;

then since the minute-hand goes 12 times round, whilst the hour-hand goes once, we have this proportion,

$$12 : 1 :: 5+x : x,$$

and (*Alg.* 180.) $11 : 1 :: 5 : x$;

$$\therefore 11x=5,$$

$$\text{and } x=\tfrac{5}{11}=27'.16\tfrac{4}{11}''.$$

---

61. Divide the number 49 into two such parts that the greater increased by 6 may be to the less diminished by 11 as 9 to 2.

Let $x=$ the greater;

$\therefore 49-x=$ the less,

and $x+6 : 38-x :: 9 : 2$;

$\therefore$ (*Alg.* 177. 179.) $x+6 : 44 :: 9 : 11$,

and (*Alg.* 186.) $x+6 : 4 :: 9 : 1$;

$\therefore x+6=36$,

and $x=30$;

$\therefore$ the parts are 30 and 19.

---

62. *A*, *B*, and *C* make a joint stock; *A* puts in £.60 less than *B*, and £.68 more than *C*; and the sum of the shares of *A* and *B* is to the sum of the shares of *B* and *C* as 5 to 4. What did each put in?

Let $x=$ what *A* put in;

$\therefore x+60=$ what *B* put in,

and $x-68=$ what *C* put in;

then $2x+60 : 2x-8 :: 5 : 4$,

and (*Alg.* 184.) $x+30 : x-4 :: 5 : 4$;

$\therefore$ (*Alg.* 180.) $34 : x - 4 :: 1 : 4$;
$\therefore 136 = x - 4$,
and $x = 140$;
$\therefore$ they put in, £.140, £.200, and £.72, respectively.

63. It is required to divide the number 34 into two such parts, that the difference between the greater and 18, shall be to the difference between 18 and the less :: 2 : 3.

Let x = the greater;
$\therefore 34 - x$ = the less,
and $x - 18 : x - 16 :: 2 : 3$;
$\therefore$ (*Alg.* 177. 180.) $x - 18 : 2 :: 2 : 1$;
$\therefore x - 18 = 4$,
and $x = 22$;
$\therefore$ the parts are 22 and 12.

64. A Bookseller sells two books, one containing 100 sheets for 10 shillings, the other containing 50 sheets, for 6 shillings, each being bound at the same price. What was that price?

Let x = the price;
then $10 - x : 6 - x :: 100 : 50 :: 2 : 1$;
$\therefore$ (*Alg.* 180.) $4 : 6 - x :: 1 : 1$;
$\therefore 4 = 6 - x$;
by transposition, $x = 2$.

65. A man wished to inclose a piece of ground with palisadoes, and found that if he set them a foot asunder, he should have too few by 150; but if he set them a yard asunder, he should have too many by 70. How many had he?

Let x = the number;
then $x - 70 : x + 150 :: 1 : 3$,
and (*Alg.* 181.) $x - 70 : 220 :: 1 : 2$,

$$\text{or } x - 70 : 110 :: 1 : 1;$$
$$\therefore x - 70 = 110,$$
$$\text{and } x = 180.$$

66. A footman, who contracted for £. 8 a year, and a livery suit, was turned away at the end of 7 months, and received only £.2. 3*s*. 4*d*. and his livery. What was its value?

$$\text{Let } x = \text{its value, in pounds;}$$

$$\text{then } 12 : 7 :: \left(8 + x : \frac{13}{6} + x ::\right) 48 + 6x : 13 + 6x;$$

$$\therefore (Alg.\ 180.)\ 5 : 7 :: 35 : 13 + 6x,$$
$$\text{and } 1 : 7 :: 7 : 13 + 6x;$$
$$\therefore 13 + 6x = 49;$$
$$\text{by transposition, } 6x = 36,$$
$$\text{and } x = 6.$$

67. What number is that to which if 1, 5, and 13, be severally added, the first sum shall be to the second, as the second to the third?

$$\text{Let } x = \text{the number required;}$$
$$\text{then } x + 1 : x + 5 :: x + 5 : x + 13$$
$$\therefore (Alg.\ 181.)\ x + 1 : 4 :: x + 5 : 8,$$
$$\text{and alt}^{\text{do}}.\ x + 1 : x + 5 :: 4 : 8 :: 1 : 2;$$
$$\therefore x + 1 : 4 :: 1 : 1,$$
$$x + 1 = 4,$$
$$\text{and } x = 3.$$

68. A Landlord let his farm for £.10 a year in money and a corn-rent. When corn sold at 10*s*. a bushel, he received at the rate of 10 shillings an acre for his land; but when it sold at 13*s*. 6*d*. a bushel, 13 shillings an acre. Of how many bushels did the corn-rent consist?

Let x = the number of bushels;
then $10x + 200$ = the annual income (in shillings);
and $\therefore x + 20$ = the number of acres;

also in the second case $\frac{27x + 400}{26}$ = the number of acres;

$$\therefore \frac{27x + 400}{26} = x + 20,$$

and $27x + 400 = 26x + 520$;
by transposition, $x = 120$.

69. When the price of a bushel of barley wanted but 3*d*. to be to the price of a bushel of oats as 8 to 5, nine bushels of oats were received as an equivalent for four bushels of barley, and 7*s*. 6*d*. in money. What was the price of a bushel of each?

Let $5x$ = the price of a bushel of oats;
$\therefore 8x - 3$ = the price of a bushel of barley;

$$45x = 32x - 12 + 90;$$

by transposition, $13x = 78$,
and $x = 6$;
$\therefore$ the price of a bushel of oats = 30*d*.
and the price of a bushel of barley = 45*d*.

70. A Countryman had two flocks of sheep, the smaller consisting entirely of ewes, each of which brought him 2 lambs. On counting them he found that the number of lambs was equal to the difference between the two flocks. If all his sheep had been ewes, and brought forth 3 lambs apiece, his stock would have been 432. Required the number in each flock.

Let x = the number in the less;
$\therefore 2x$ = the number of lambs this flock produced = the difference of the flocks,

and $3x$ = the number in the larger flock;

$\therefore 4x + 3 \times 4x = 4 \times 4x = 432,$

and $x = 27$;

$\therefore$ 27 and 81, are the numbers required.

71. A market-woman bought a certain number of eggs at two a penny, and as many at three a penny, and sold them out at the rate of five for two-pence; after which she found that instead of making her money again as she expected, she lost four-pence by them. How many eggs of each sort had she?

Let x = the number required;

then $2 : x :: 1 :$ the price of x eggs at 2 a penny $= \frac{x}{2}$;

in the same way, $\frac{x}{3}$ = the price of x eggs at 3 a penny,

and $5 : 2x :: 2 :$ the price at which she sold all, $= \frac{4x}{5}$;

$$\therefore \frac{4x}{5} + 4 = \frac{x}{2} + \frac{x}{3},$$

and $24x + 120 = (15x + 10x =) \, 25x$;

by transposition, $x = 120$.

72. A man and his wife did usually drink out a vessel of beer in 12 days: but when the man was out, the vessel lasted the woman 30 days. In how many days would the man alone drink it out?

Let x = the number of days required;

then $x : 12 :: 1 :$ the part drank by the man

in 12 days $= \frac{12}{x}$,

and 30 : 12 :: 1 : the part drank by the woman

$$\text{in 12 days} = \frac{12}{30} = \frac{2}{5};$$

$$\therefore \frac{12}{x} + \frac{2}{5} = 1,$$

$$\text{and } 60 + 2x = 5x;$$

$$\text{by transposition, } 60 = 3x,$$

$$\text{and } 20 = x.$$

73. A cistern into which water was let by 2 cocks *A* and *B*, will be filled by them both running together in 12 hours, and by the cock *A* alone in 20 hours. In what time will it be filled by the cock *B* alone?

Let x = the number of hours;

then x : 12 :: 1 : the quantity supplied by *B*

$$\text{in 12 hours} = \frac{12}{x}.$$

In the same way, the quantity supplied by *A*

$$\text{in 12 hours} = \frac{3}{5};$$

$$\therefore \frac{12}{x} + \frac{3}{5} = 1,$$

$$\text{and } 60 + 3x = 5x;$$

$$\text{by transposition, } 60 = 2x,$$

$$\text{and } 30 = x.$$

74. The hold of a ship contained 442 gallons of water. This was emptied out by two buckets, the greater of which, holding twice as much as the other, was emptied twice in three minutes, but the less three times in two minutes; and the whole time of emptying was 12 minutes. Required the size of each.

Let x = the number of gallons the less held;

$\therefore 2x$ = the number the greater held;

and $4x$=the quan^y. thrown out by the greater in 3 min^s.;

$\therefore$ (3 : 12 ::) 1 : 4 :: $4x$: the quantity thrown out in 12 minutes $= 16x$.

In the same manner the quantity thrown out by the less in 12 minutes $= 18x$;

$$\therefore (18x + 16x =)\ 34x = 442,$$

$$\text{and } x = 13\ ;$$

$\therefore$ the less held 13, and the greater 26 gallons.

---

75. A hare, 50 of her leaps before a greyhound, takes 4 leaps to the greyhound's three; but two of the greyhound's leaps are as much as three of the hare's. How many leaps must the greyhound take to catch the hare?

Let $3x$ = the number of leaps the greyhound must take;

$\therefore 4x$ = the number the hare takes in the same time;

$\therefore 4x + 50$ = the whole number she takes,

$$\text{and } 2 : 3 :: 3x : 4x + 50\ ;$$

$$\therefore 9x = 8x + 100\ ;$$

by transposition, $x = 100$,

and the greyhound must take 300 leaps.

---

76. If 10 apples cost a penny, and 25 pears cost two-pence, and I buy 100 apples and pears for nine-pence half-penny, how many of each shall I have?

Let x = the number of apples;

$\therefore 100 - x$ = the number of pears;

and 10 : x :: 1 : the price of x apples $= \dfrac{x}{10}$.

In the same manner the price of the pears $= \dfrac{200 - 2x}{25}$;

$$\therefore \frac{x}{10} + \frac{200 - 2x}{25} = \frac{19}{2},$$

and $5x + 400 - 4x = 475$;

by transposition, $x = 75$;

$\therefore$ the number of apples is 75, and pears 25.

---

77. A person has two sorts of wine, one worth 20 pence a quart, and the other 12 pence; from which he would mix a quart to be worth 14 pence. How much of each must he take?

Let $x =$ the quantity of the first, the whole quart being represented by unity;

$\therefore 1 - x =$ the quantity of the second;

also $20x =$ the value of the first,

and $12 - 12x =$ the value of the second;

$\therefore 20x + 12 - 12x = 14$;

and by transposition, $8x = 2$;

$\therefore x = \frac{1}{4}$;

$\therefore$ he must take $\frac{1}{4}$ of the first, and $\frac{3}{4}$ of the second.

---

78. A person engaged to reap a field of 35 acres, consisting partly of wheat, and partly of rye. For every acre of rye he received 5 shillings; and what he received for an acre of wheat augmented by one shilling, is to what he received for an acre of rye as 7 to 3. For his whole labour he received £.13. Required the number of acres of each sort.

Let $x =$ the number of acres of wheat;

$\therefore 35 - x =$ the number of acres of rye;

and $175 - 5x =$ the price of reaping them.

Now $3 : 7 :: 5 : 1 +$ the price of reaping an acre of wheat $= \frac{35}{3}$;

$\therefore$ the price of reaping an acre of wheat $= \frac{32}{3}$,

and the price of reaping all the wheat $= \frac{32x}{3}$;

$$\therefore \frac{32x}{3} + 175 - 5x = 260,$$

$$\text{and } 32x + 525 - 15x = 780;$$

$$\text{by transposition, } 17x = 255,$$

$$\text{and } x = 15;$$

$\therefore$ there were 15 acres of wheat, and 20 of rye.

79. Two pieces of cloth of equal goodness, but of different lengths, were bought, the one for £.5, the other for £.6. 10*s*. Now if the lengths of both pieces were increased by 10, the numbers resulting would be in the proportion of 5 to 6. How long was each piece, and how much did they cost a yard?

Let x = the number of 10 shillings that each yard cost,

then $\frac{10}{x}$ = the length of the least,

and $\frac{13}{x}$ = the length of the longest;

$$\text{also } \frac{10}{x} + 10 : \frac{13}{x} + 10 :: 5 : 6;$$

$$\therefore (\textit{Alg.}\ 181.)\ 10 . \left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 5 : 1,$$

$$\text{and } (\textit{Alg.}\ 186.)\ 2 . \left(\frac{1}{x} + 1\right) : \frac{3}{x} :: 1 : 1;$$

$$\therefore 2 . \frac{x+1}{x} = \frac{3}{x},$$

$$\text{and } 2x + 2 = 3;$$

$$\text{by transposition, } 2x = 1,$$

$$\text{and } x = \frac{1}{2};$$

$\therefore$ the price is 5*s*. and the lengths are 20 and 26 yards.

80. A General, whose horse was $\frac{1}{3}$ of his foot, after a defeat found, that before the battle $\frac{1}{12} - 120$ of his foot, and $\frac{1}{12} + 120$ of his horse had deserted; $\frac{1}{4}$ of his whole army was in garrison; and $\frac{3}{8}$ remained, the rest being either taken prisoners or slain. Now 300 + the number slain = $\frac{1}{2}$ the foot he had at first. Of how many did his whole army consist?

Let x = the number of horse;
$\therefore 3x$ = the number of foot,
and $4x$ = the whole army;
also $\frac{3x}{2} - 300$ = the number slain;

$$\therefore \frac{x}{4} - 120 + \frac{x}{12} + 120 + x + \frac{3x}{2} + \frac{3x}{2} - 300 = 4x,$$

and $3x + x + 48x - 3600 = 48x$;
by transposition, $4x = 3600$,
and $x = 900$;

$\therefore$ the whole army consisted of 3600 men; viz. 900 horse, and 2700 foot.

81. Two persons *A* and *B* have both the same annual income. *A* lays by $\frac{1}{5}$th of his; but *B* by spending £.80 per annum more than *A*, at the end of 4 years finds himself £.220 in debt. What did each receive and expend annually?

Let $5x$ = their annual income;
$\therefore 4x$ = *A*'s annual expenditure,
and $4x + 80$ = *B*'s annual expenditure;
$\therefore (4x + 80 - 5x =)\ 80 - x$ = the debt *B* annually incurs;

$\therefore 320 - 4x = 220$;
by transposition, $4x = 100$,
and $x = 25$;

$\therefore$ their annual income is £.125;
A's annual expenditure is £.100, and *B*'s £.180.

82. A person at play won twice as much as he began with, and then lost 16 shillings. After this he lost four-fifths of what remained, and then won as much as he began with, and counting his money, found he had 80 shillings. What sum did he begin with?

Let x = the number of shillings he began with;
then $3x$ = the sum he had, after winning $2x$,
and $3x - 16$ = the sum remaining after the next loss.

Now since he lost $\frac{4}{5}$ of this, $\frac{3x-16}{5}$ = the sum remaining;

$$\therefore \frac{3x - 16}{5} + x = 80,$$

$$\text{and } 3x - 16 + 5x = 400;$$

$$\text{by transposition, } 8x = 416,$$

$$\text{and } x = 52.$$

83. Having lost one-third of my money at play, I won 3 times as much as I had left, half as much money as I began with, and £.50; and then found I had as much above £.100, as the sum I began with was below £.100. What sum did I begin with?

Let $6x$ = the number of pounds required;
then $4x$ = the sum remaining after $\frac{1}{3}$ was lost,
and $12x + 3x + 50$ = the sum afterwards won;
$\therefore (12x + 3x + 50 + 4x =)\ 19x + 50$ = the whole sum he had.

$$\text{and } 19x + 50 - 100 = 100 - 6x;$$

$$\text{by transposition, } 25x = 150,$$

$$\text{and } x = 6;$$

$\therefore$ he began with 36 pounds.

84. *A* and *B* began to pay their debts. *A*'s money was at first two-thirds of *B*'s; but after *A* had paid £.1 less than two-thirds of his money, and *B* £.1 more than seven-eighths of his, it was found that *B* had only half as much as *A* had left. What sum had each at first?

Let $2x$ and $3x$ = the sums *A* and *B* had respectively,

$$\left.\begin{array}{r}\text{then after payment, } A \text{ had } \frac{2x}{3} + 1 \\ \text{and } B \text{ had } \frac{3x}{8} - 1\end{array}\right\} \text{remaining;}$$

$$\therefore \frac{2x}{3} + 1 = \frac{3x}{4} - 2;$$

$$\therefore 8x + 12 = 9x - 24;$$

by transposition, $36 = x$;

$\therefore$ *A* had £.72, and *B* had £.108.

---

85. It is required to divide the number 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other.

Let $2x$ = the first part;

$\therefore x = \frac{1}{3}$ part of the second, and $3x$ = the second;

also $x = \frac{1}{4}$ part of the third, and $\therefore 4x$ = the third;

$\therefore (2x + 3x + 4x =)\ 9x = 36,$

and $x = 4$;

$\therefore$ the parts are 8, 12, and 16.

---

86. Divide the number 116 into four such parts, that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be the same.

Let x = the third;

$\therefore 3x$ = half the fourth,

and $6x$ = the fourth;

whence $3x + 4$ = the second,

and $3x - 5$ = the first;

$$\therefore 3x - 5 + 3x + 4 + x + 6x = 116;$$

by transposition, $13x = 117$,
and $x = 9$;
$\therefore$ the parts are 22, 31, 9, and 54.

87. A Gentleman had some of his horses at grass at 3 shillings each a week, and the rest at livery stables at 10 shillings each a week. The horses in the stables cost him twice as much a week as the horses at grass. But he finds that if he had sent 3 horses to grass out of the stables, the expence of the stables would have been only 6 shillings a week more than the grass. How many horses had he?

Let x = the number of horses at grass;
$\therefore 3x$ = the weekly expence of these,
and $6x$ = - - - - - of horses in stables;

$$\therefore \frac{6x}{10} = \text{their number};$$

also $6x - 30$ = the expence of the stables after 3 horses were sent out to grass,
and $3x + 9$ = the expense of the horses at grass, when 3 more were added;

$$\therefore 6x - 30 \; (= 3x + 9 + 6) = 3x + 15;$$

by transposition, $3x = 45$,
and $x = 15$;
$\therefore$ there were 15 at grass, and 9 in the stables.

88. A silversmith received in payment for a certain weight of wrought plate, the price of which was £.10, the same weight of unwrought plate, and £.3. 15s. besides. At another time he exchanged 12 oz. of wrought plate of the same workmanship as before for 8 oz. of unwrought (for which he allowed the same price as before), and £.2. 16s. in money. What was the price of wrought plate per ounce, and the weight of the first sold?

Let x = the number of ounces;

$\therefore \frac{200}{x}$ = the price of an oz. wrought,

and $\frac{125}{x}$ = the price of an oz. unwrought;

$$\therefore \frac{2400}{x} = \frac{1000}{x} + 56;$$

by transposition, $\frac{1400}{x} = 56$,

and $\frac{25}{x} = 1$;

$\therefore x = 25$;

whence there were 25 ounces; and the price was 8 shillings per oz.

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89. In changing a bill of £.85 into guineas and shillings (the number of shillings being ¼ number of guineas) on examination they all proved adulterated below the standard value, to the amount in the whole of £.8. 5*s.* To make up the deficiency, nine more such guineas were paid; and four such shillings and three good ones returned. Required the number and average value of the guineas and shillings paid at first.

£.85 = 85*s.* × 20 = the N°. of shillings + 4 × 21 × N°. of shil^s.

= 85 × number;

$\therefore$ the number of shillings = 20,

and the number of guineas = 80;

Let x = the value of an adulterated guinea;

$\therefore$ (1700 − 165 =) 1535 = 80x + 20 × value of an adulterated shilling,

and $\therefore$ the value of an adulterated shilling $= \frac{1535 - 80x}{20} =$

$$\frac{307 - 16x}{4};$$

$$\therefore\ 165 = 9x - 3 - 307 + 16x = 25x - 310;$$

by transposition, $475 = 25x$,

and $19 = x$;

$\therefore$ the value of an adulterated guinea $=$ 19*s*. and the value of an adulterated shilling $= \frac{307 - 16x}{4} = \frac{3}{4} = 9d.$

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90. Before noon, a clock which is too fast, and points to afternoon time is put back five hours and forty minutes; and it is observed that the time before shewn is to the true time as 29 to 105. Required the true time.

Let $x =$ the time the clock pointed to;

then $x : x + 6\frac{1}{3} :: 29 : 105$;

(*Alg.* 181.) $x : \frac{19}{3} :: 29 : 76$,

and $x : \frac{1}{3} :: 29 : 4$;

whence $x = \frac{29}{12} = 2^h\ 25'$;

if $\therefore$ this be added to $6^h\ 20'$ the true time is $8^h\ 45'$, or $15'$ before 9.

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91. The crew of a ship consisted of her complement of sailors and a number of soldiers. Now there were 22 seamen to every 3 guns and 10 over. Also the whole number of hands, was five times the number of soldiers and guns together. But after an engagement, in which the slain were one-fourth of the survivors, there wanted 5, to be 13 men to every 2 guns. Required the number of guns, soldiers, and sailors.

Let $3x =$ the number of guns;

then $22x + 10 =$ the number of seamen,

and since, seamen $+$ soldiers $= 5$. soldiers $+ 15x$;

$\therefore$ the number of soldiers $= \frac{1}{4}(22x + 10 - 15x) = \dfrac{7x+10}{4}$

and the complement $= \dfrac{7x+10}{4} + 22x + 10 = \dfrac{95x+50}{4}$,

and the survivors were $\frac{4}{5}\left(\dfrac{95x+50}{4}\right) = 19x + 10$;

$$\therefore \frac{3x}{2}.13 - 5 = 19x + 10;$$

by transposition, $\dfrac{x}{2} = 15$,

and $x = 30$;

$\therefore$ the number of guns $= 90$;

the number of seamen $= 30 \times 22 + 10 = 670$,

and the number of soldiers $= \dfrac{7 \times 30 + 10}{4} = 55$.

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92. A shepherd, in time of war, was plundered by a party of soldiers, who took $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep, another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep; then a third party took $\frac{1}{2}$ of what now remained; and $\frac{1}{2}$ of a sheep. After which he had but 25 sheep left. How many had he at first?

Let $x =$ the number he had at first;

then $\dfrac{x+1}{4} =$ the number the first party took away,

and $\therefore \dfrac{3x-1}{4} =$ the number remaining.

Now the second party took away $\frac{1}{3}$ of these $+ \frac{1}{3}$ of a sheep;

$\therefore$ there remained $\frac{2}{3}\left(\dfrac{3x-1}{4}\right) - \frac{1}{3} = \dfrac{3x-1}{6} - \frac{1}{3} = \dfrac{3x-3}{6} = \dfrac{x-1}{2}$

then the third party took away half of these $+ \frac{1}{2}$ of a sheep;

$\therefore$ there remained $\dfrac{x-1}{4} - \frac{1}{2} = \dfrac{x-3}{4}$;

$$\text{whence } \frac{x-3}{4} = 25,$$

$$\text{and } x-3 = 100;$$

$$\text{by transposition, } x = 103.$$

---

93. A man being at play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had, and won two shillings; lastly he lost $\frac{1}{7}$ of what he then had; this done he had but 12 shillings left. What had he at first?

Let $4x =$ the number of shillings required;
then after the first loss he had $3x$, and afterwards $3x+3$;
after the second loss he had $\dfrac{2}{3} \cdot (3x+3) = 2x+2$, and afterwards $2x+4$.

Having lost $\frac{1}{7}$ of this, he had $\frac{6}{7} \cdot (2x+4) = \dfrac{12x+24}{7}$,

$$\text{and } \therefore \frac{12x+24}{7} = 12,$$

$$\text{and } 12x+24 = 7 \times 12;$$

$$\text{by transposition, } 12x = 5 \times 12,$$

$$\text{and } x = 5;$$

$\therefore$ he had at first 20 shillings.

---

94. A Trader maintained himself for 3 years at the expence of £.50 a year; and in each of those years, augmented that part of his stock which was not so expended by $\frac{1}{3}$ thereof. At the end of the third year his original stock was doubled. What was that stock?

Let $x =$ the number of pounds required;

then $x-50$ = the sum not expended; and with this he traded;

$\therefore \frac{x-50}{3}$ = his gain the first year,

and $\frac{4}{3}.(x-50)$ = the sum he had at the end of the first year;

$\therefore \frac{4x-200}{3}-50=\frac{4x-350}{3}$ = the sum he traded with the second year;

$\therefore \frac{4}{3}.\frac{4x-350}{3}=\frac{16x-1400}{9}$ = the sum he had at the end of the second year;

and $\frac{16x-1400}{9}-50=\frac{16x-1850}{9}$ = the sum he traded with the third year;

$\therefore \frac{4}{3}.\frac{16x-1850}{9}$ = the sum he had at the end of the third year;

$$\text{whence } \frac{4}{3}.\frac{16x-1850}{9}=2x,$$

$$\text{and } 32x-3700=27x;$$

$$\text{by transposition, } 5x=3700,$$

$$\text{and } x=740.$$

95. A merchant buys a cask of brandy for £.48, and sells a quantity exceeding three-fourths of the whole by 2 gallons at a profit of £.25 per cent. He afterwards sells the remainder at such a price as to clear £.60 per cent. by the whole transaction; and, had he sold the whole quantity at the latter price, he would have gained £.175 per cent. Required the number of gallons contained in the cask.

Let $4x$ = the number of gallons;

then $\frac{12}{x}$ = the original price per gallon (in pounds,)

and $100 : 125 :: \frac{12}{x}$: the first price of sale $= \frac{15}{x}$,

and $100 : 275 :: \frac{12}{x}$: the latter price of sale $= \frac{33}{x}$;

$\therefore (3x+2) \cdot \frac{15}{x} + (x-2) \cdot \frac{33}{x} - 48 =$ whole gain $= 30 - \frac{36}{x}$;

hence $100 : 60 :: 48 : 30 - \frac{36}{x}$,

or $5 : 3 :: 8 : 5 - \frac{6}{x}$;

$\therefore 24 = 25 - \frac{30}{x}$;

by transposition, $\frac{30}{x} = 1$,

and $\therefore x = 30$,

and the number required $= 4x = 120$ gallons.

96. Water flows uniformly into a cistern, capable of containing 720 gallons, through a pipe; and at the same time is discharged by a pump, worked by three men, who take four strokes in a minute; but this not being sufficient, the cistern becomes full in 6 hours; they therefore now put in another pump, of such power that the quantity discharged at one stroke by this pump is to the quantity discharged at one stroke by the former :: 2 : 3; but being obliged to detach one of their number to work the pump, the former pump makes only 10 strokes in 3 minutes, and the latter 5 strokes in 2 minutes; by which means the cistern is emptied in 12 hours. How much water was discharged by each pump at one stroke? and how much flowed in through the pipe in one minute?

Let $3x$ = the number of gallons discharged by the first pump at one stroke;

$\therefore 2x$ = the number discharged by the second,

and $12x$ = the quantity discharged by the first in one second, when 3 men worked;

$\therefore 6 \times 60 \times 12x$ = the quantity discharged in six hours;

$\therefore 6 \times 60 \times 12x + 720 = 720 \cdot (6x+1)$ = the quantity introduced through the pipe in that time.

Now when the additional pump is worked,

$10x$ = quantity discharged by the first in one minute;

and $5x$ = the quantity discharged by the second in one minute;

$\therefore 15x \times 12 \times 60$ = the whole quantity discharged in 12 hours;

$$\therefore 15x \times 720 = 720 \times 2 \cdot (6x+1) + 720;$$

$$\text{or } 15x = 12x + 3;$$

$$\text{by transposition, } 3x = 3,$$

$$\text{and } x = 1;$$

$\therefore$ the first discharged 3 gallons, and the second 2, at one stroke; and the quantity introduced by the pipe in one minute $= \dfrac{720 \times (6x+1)}{6 \times 60} = 2 \times 7 = 14$ gallons.

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97. A poor man with a wife and seven children, found during a scarcity that he could only earn sufficient to procure $\frac{1}{4}$ of a white loaf of bread *per* day for each of his family, himself included. He therefore applied to the parish officers for assistance, by whom being allowed a daily sum $= \frac{1}{2}$ his earnings, and mixed bread being made by order of Parliament, which was cheaper than white in the proportion of 4 to 5, he was now enabled to procure $\frac{1}{3}$ of a mixed loaf *per* day for each of the family (himself still included) and had 1*s.* $7\frac{1}{2}$*d.* over. Required the sum allowed him by the parish.

Let x = the price of a white loaf (in pence;)

$\therefore \dfrac{9x}{4}$ = what he earned,

and $\frac{9x}{8}$ = what the parish allowed him;

also $\frac{4x}{5}$ = the price of a mixed loaf;

$$\therefore \frac{12x}{5} + \frac{39}{2} = \left(\frac{9x}{4} + \frac{9x}{8} =\right) \frac{27x}{8},$$

$$\text{and } 96x + 780 = 135x;$$

$$\text{by transposition, } 780 = 39x,$$

$$\text{and } 20 = x;$$

whence it appears that he earned 45 pence, and had $22\frac{1}{2}d.$ allowed him by the parish.

SECT. VII.

Examples of the Solution of Problems producing Simple Equations involving two unknown Quantities.

1. AFTER A had won four shillings of B, he had only half as many shillings as B had left. But had B won six shillings of A, then he would have had three times as many as A would have had left. How many had each?

Let x = the number of shillings A had,
and y = the number B had;

$$\text{then } y - 4 = 2x + 8,$$
$$\text{and } y + 6 = 3x - 18;$$
$$\therefore \text{ by subtraction, } 10 = x - 26,$$
$$\text{and by transposition, } 36 = x,$$

and $y - 4 = 80$; $\therefore y = 84$; $\therefore$ A had 36 and B 84.

2. A person bought a quantity of brandy and rum for £.19. 4*s*. and gave for the brandy 9 shillings, and rum 6 shillings *per* bottle. He found however that he could have bought as many bottles of rum as he now had of brandy, and as many of brandy as he now had of rum for £.1. 13*s*. less. How much was bought?

Let x = the number of bottles of brandy,
and y = the number of bottles of rum;

$$\text{then } 9x + 6y = 384,$$
$$\text{and } 6x + 9y = 351;$$
$$\therefore \text{ by addition, } 15x + 15y = 735,$$

$$\begin{aligned} \text{and } x + y &= 49. \\ \text{But since } 3x + 2y &= 128, \\ \text{and } 2x + 2y &= 98; \\ \therefore \text{ by subtraction, } x &= 30, \\ \text{and } y = 49 - x &= 19; \end{aligned}$$

$\therefore$ he bought 30 bottles of brandy, and 19 of rum.

---

3. What fraction is that, to the numerator of which if 4 be added, the value is one-half, but if 7 be added to the denominator, its value is one-fifth?

$$\text{Let } \frac{x}{y} = \text{the fraction required;}$$

$$\text{then, } \frac{x+4}{y} = \tfrac{1}{2}, \text{ and } \therefore 2x + 8 = y;$$

$$\text{also } \frac{x}{y+7} = \tfrac{1}{5}, \text{ and } \therefore 5x = y + 7;$$

$$\begin{aligned} \text{by subtraction, } 3x - 8 &= 7; \\ \text{by transposition, } 3x &= 15, \\ \text{and } x &= 5; \\ \therefore y = 2x + 8 &= 18, \end{aligned}$$

$$\text{and the fraction is } \frac{5}{18}.$$

---

4. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

$$\begin{aligned} &\text{Let } x \text{ and } y = \text{the numbers;} \\ &\text{then, } x : y :: x + y : 42, \\ &\text{and } x : y :: x - y : 6. \end{aligned}$$

But ratios which are equal to the same ratio are equal to each other;

$\therefore x + y : 42 :: x - y : 6,$

alt^{do}. $x + y : x - y :: (42 : 6 ::)\ 7 : 1;$

$\therefore$ (Add. & Sub.) $2x : 2y :: 8 : 6,$

and $x : y :: 4 : 3;$

$$\therefore x = \frac{4y}{3},$$

$$\text{and } 4 : 3 :: \frac{y}{3} : 6;$$

$$\therefore \frac{y}{3} = 8, \text{ and } y = 24,$$

$$\text{and } x = \frac{4y}{3} = 32;$$

$\therefore$ the numbers are 32 and 24.

---

5. What two numbers are those, whose difference, sum, and product, are as the numbers 2, 3, and 5, respectively?

Let x and y = the numbers;

then $x - y : x + y :: 2 : 3;$

$\therefore 2x : 2y :: 5 : 1,$

or $x : y :: 5 : 1;$

also $x + y : xy :: 3 : 5,$

or $6y : xy :: 3 : 5;$

$\therefore 2 : x :: 1 : 5,$

and $\therefore x = 10,$

$$\text{and } y = \frac{x}{5} = 2;$$

$\therefore$ the numbers are 10 and 2.

---

6. *A* and *B* playing at bowls, says *A* to *B*, If you will give me a guinea, I will bet you half a crown to eighteen pence on

each game, and will play 36 games together. *B* won his guinea back again, and £.1. 17*s*. besides. How many games did each win?

Let x = the number of games *A* won,
and y = the number *B* won;

$$\therefore y + x = 36,$$
$$\text{and } 3y + 3x = 108;$$
$$\text{but } 5y - 3x = 116;$$
$$\therefore \text{by addition, } 8y = 224,$$
$$\text{and } y = 28;$$
$$\therefore x = 36 - y = 8;$$

∴ *A* won 8, and *B* 28 games.

---

7. A person exchanged 12 bushels of wheat for 8 bushels of barley, and £.2. 16*s*; offering at the same time to sell a certain quantity of wheat for an equal quantity of barley, and £.3. 15*s*. in money, or for £.10. in money. Required the prices of the wheat and barley per bushel.

Let x = the price of wheat per bushel, in shillings,
and y = the price of barley;

then $\frac{200}{x}$ = the number of bushels in the second offer;

$$\therefore 12x = 8y + 56,$$
$$\text{and } \frac{200}{x} \times y = 125;$$
$$\therefore 8y = 5x,$$
$$\text{and } \therefore 12x = 5x + 56;$$
$$\text{by transposition, } 7x = 56,$$
$$\text{and } x = 8;$$
$$\therefore y = 5;$$

∴ the prices of wheat and barley *per* bushel were 8 and 5 shillings, respectively.

8. A vintner sold at one time 20 dozen of port wine, and 30 of sherry, and for the whole received £.120; and at another time sold 30 dozen of port, and 25 of sherry at the same prices as before, and for the whole received £.140. What was the price of a dozen of each sort of wine?

Let x = the price of a dozen of port,
and y = - - - - - - - of sherry;

$$\therefore\ 20x + 30y = 120, \text{ or } 2x + 3y = 12,$$
$$\text{and } 30x + 25y = 140, \text{ or } 6x + 5y = 28.$$

Multiplying the first equation by 3;

$$6x + 9y = 36,$$
$$\text{but } 6x + 5y = 28;$$
$$\therefore \text{ by subtraction, } 4y = 8,$$
$$\text{and } y = 2;$$
$$\text{whence } 2x = 12 - 3y = 12 - 6 = 6,$$
$$\text{and } x = 3;$$

∴ the prices of port and sherry *per* dozen were £.3 and £.2, respectively.

---

9. *A* and *B* severally cut packs of cards, so as to cut off less than they left. Now the number of cards left by *A* added to the number cut off by *B* make 50; also the number of cards left by both exceed the number cut off, by 64. How many did each cut off?

Let x = the number cut off by *A*;
∴ $52 - x$ = the number left by him.
Let y = the number cut off by *B*;
∴ $52 - y$ = the number left by him;
∴ $x + y$ = the whole number cut off,
and $104 - (x + y)$ = the whole number left,

$$\text{whence } 104 - 2.(x + y) = 64;$$
$$\text{by transposition, } 2.(x + y) = 40,$$
$$\text{and } x + y = 20.$$

$$\text{Now } 52 - x + y = 50;$$
$$\therefore \text{ by transposition, } x - y = 2,$$
$$\text{but } x + y = 20;$$
$$\therefore \text{ by addition } 2x = 22,$$
$$\text{and } x = 11;$$
$$\text{by subtraction, } 2y = 18,$$
$$\text{and } y = 9;$$
$\therefore$ *A* cut off 11, and *B* 9.

10. A countryman, being employed by a poulterer to drive a flock of geese and turkeys to London, in order to distinguish his own from any he might meet on the road, pulled 3 feathers out of the tail of each turkey, and one out of the tail of each goose, and upon counting them, found that the number of turkeys' feathers exceeded twice those of the geese by 15. Having bought 10 geese and sold 15 turkeys by the way, he was surprised to find, as he drove them into the poulterer's yard, that the number of geese exceeded the number of turkeys in the proportion of 7 to 3. Required the number of each at first.

Let x = the number of turkeys;
$\therefore 3x$ = the number of feathers from their tails;
let y = the number of geese; and $\therefore$ of the feathers,

$$\text{and } 3x - 2y = 15.$$
$$\text{Also } y + 10 : x - 15 :: 7 : 3;$$
$$\therefore 3y + 30 = 7x - 105;$$
$$\text{by transposition, } 7x - 3y = 135,$$
$$\text{and } 14x - 6y = 270;$$
$$\text{but from the first equation } 9x - 6y = 45;$$
$$\therefore \text{ by subtraction, } 5x = 225,$$
$$\text{and } x = 45;$$
$$\therefore 2y = 3x - 15 = 135 - 15 = 120,$$
$$\text{and } y = 60;$$
$\therefore$ there were 45 turkeys, and 60 geese.

11. A farmer with 28 bushels of barley at 2*s.* 4*d.* a bushel, would mix rye at 3 shillings *per* bushel, and wheat at 4 shillings *per* bushel, so that the whole mixture may consist of 100 bushels, and be worth 3*s.* 4*d.* *per* bushel. How many bushels of rye, and how many of wheat, must he mix with the barley?

Let x = the number of bushels of rye,
and y = the number of wheat;
then the value of the barley = 196 (four-pences),
of the wheat = $12y$,
of the rye = $9x$;

$$\therefore 196 + 9x + 12y = 1000;$$

by transposition, $9x + 12y = 804$,
and $3x + 4y = 268$.

Now $x + y + 28 = 100$,
and by transposition, $x + y = 72$;

$$\therefore 3x + 3y = 216,$$

but $3x + 4y = 268$;

$\therefore$ by subtraction, $y = 52$,
and $x = 72 - y = 20$.

Hence he must mix 20 bushels of rye, and 52 of wheat.

---

12. *A* and *B* speculate with different sums; *A* gains £150, *B* loses £.50, and now *A*'s stock is to *B*'s as 3 to 2. But had *A* lost £.50, and *B* gained £.100, then *A*'s stock would have been to *B*'s as 5 to 9. What was the stock of each?

Let x = *A*'s stock,
and y = *B*'s;
then $x + 150 : y - 50 :: 3 : 2$;

$$\therefore 2x + 300 = 3y - 150,$$

and by transposition, $3y - 2x = 450$;
also $x - 50 : y + 100 :: 5 : 9$;

$$\therefore 9x - 450 = 5y + 500;$$

by transposition, $9x - 5y = 950$;

multiplying this equation by 3, and that found above by 5,

$$27x - 15y = 2850,$$
$$\text{and } 15y - 10x = 2250;$$

$\therefore$ by addition, $17x = 5100,$

and $x = 300;$

$\therefore 3y = 2x + 450 = 1050,$

and $y = 350;$

$\therefore$ *A*'s was £.300, and *B*'s £.350.

13. A merchant having mixed a certain number of gallons of brandy and water, found that if he had mixed 6 gallons more of each, he would have put into the mixture 7 gallons of brandy for every 6 of water; but if he had mixed 6 less of each, he would have put in 6 gallons of brandy for every 5 of water. How many of each did he mix?

Let x = the number of gallons of brandy,
and y = the number of gallons of water;

then $x + 6 : y + 6 :: 7 : 6;$

$\therefore$ (*Alg.* 181.) $x + 6 : x - y :: 7 : 1,$

but $x - 6 : y - 6 :: 6 : 5,$

and $\therefore x - y : x - 6 :: 1 : 6,$

and since $x + 6 : x - y :: 7 : 1;$

$\therefore$ *ex æquali*, $x + 6 : x - 6 :: 7 : 6,$

and $2x : 12 :: 13 : 1,$

or $x : 6 :: 13 : 1;$

$\therefore x = 78;$

whence $84 : y + 6 :: 7 : 6,$

or $12 : y + 6 :: 1 : 6;$

$\therefore y + 6 = 72,$

and $y = 66;$

$\therefore$ he mixed 78 gallons of brandy with 66 of water.

14. A person had a bag of money worth £.93; but a servant having robbed him of one-sixth of his moidores, and three-

fifths of his guineas, left him only £.54. 15*s.* How many moidores and guineas had he at first.

Let x = the number of guineas,
and y = the number of moidores;

$$\text{then } \frac{45\,y}{6} + \frac{14\,x}{5} = 365,$$

$$\text{and } \therefore\ 225\,y + 84\,x = 10950\,;$$

$$\text{also } 7\,x + 9\,y = 620\,;$$

$$\therefore\ 108\,y + 84\,x = 7440\,;$$

$$\text{but since } 225\,y + 84\,x = 1095\,;$$

$$\therefore \text{ by subtraction, } 117\,y = 3510,$$

$$\text{and } y = 30\,;$$

$$\text{also } 7\,x = 620 - 9\,y = 620 - 270 = 350\,;$$

$$\therefore\ x = 50\,;$$

∴ he had 50 guineas, and 30 moidores.

15. A vintner bought 6 dozen of port wine and 3 dozen of white for 12 guineas; but the price of each afterwards falling a shilling *per* bottle, he had 20 bottles of port, and 3 dozen and 8 bottles of white more, for the same sum. What was the price of each at first?

Let x = the price of the port } *per* bottle (in shillings,)
y = - - - - white }

$$\text{then } 72\,x + 36\,y = 252 = 92\,x + 80\,y - 172,$$

$$\text{and } \therefore \text{ by transposition, } 20\,x + 44\,y = 172,$$

$$\text{or } 5x + 11\,y = 43.$$

$$\text{Now, since } 72\,x + 36\,y = 252\,;$$

$$\therefore\ 2\,x + y = 7,$$

$$\text{and } 22\,x + 11\,y = 77,$$

$$\text{but } 5\,x + 11\,y = 43\,;$$

$$\therefore \text{ by subtraction, } 17\,x = 34,$$

$$\text{and } x = 2\,;$$

$$\text{whence } y = 7 - 2\,x = 7 - 4 = 3\,;$$

∴ the price of port was 3*s*. and of white 2*s*. *per* bottle.

16. A rectangular bowling-green having been measured, it was observed, that if it were 5 feet broader, and 4 feet longer, it would contain 116 feet more: but if it were 4 feet broader, and 5 feet longer, it would contain 113 feet more. Required the length and breadth.

Let $x =$ the number of feet in length,

and $y =$ the number of feet in breadth;

then $(\overline{x+4}\,.\,\overline{y+5} =) xy + 5x + 4y + 20 = 116 + xy$,

and $\therefore 5x + 4y = 96$;

also $(\overline{x+5}\,.\,\overline{y+4} =) xy + 4x + 5y + 20 = 113 + xy$;

$\therefore 4x + 5y = 93$;

multiplying the former equation by 4, and the latter by 5,

$20x + 16y = 384$,

and $20x + 25y = 465$;

by subtraction, $9y = 81$,

and $y = 9$;

$\therefore 4x = 93 - 5y = 93 - 45 = 48$,

and $x = 12$;

∴ the length was 12 and the breadth 9 feet.

17. Find two numbers in the proportion of 5 to 7, to which two other required numbers in the proportion of 3 to 5 being repectively added, the sums shall be in the proportion of 9 to 13; and the difference of those sums = 16.

Let $5x$ and $7x =$ the two first numbers,

and $3y$ and $5y =$ the others;

then $5x + 3y : 7x + 5y :: 9 : 13$;

$\therefore 5x + 3y : 2x + 2y :: 9 : 4$;

or $5x + 3y : x + y :: 9 : 2$,

and $10x + 6y = 9x + 9y$;

by transposition, $x = 3y$;

$$\text{but } 2x + 2y = 16;$$
$$\therefore (6y + 2y =)\ 8y = 16;$$
$$\therefore y = 2,$$
$$\text{and } x = 6;$$

whence, the two first numbers are 30 and 42; the two others, 6 and 10.

18. A merchant finds that if he mixes sherry and brandy in quantities which are in the proportion of 2 to 1, he can sell the mixture at 78 shillings a dozen; but if the proportion be as 7 to 2, he must sell it at 79 shillings a dozen. Required the price of each liquor.

Let x = the price of the sherry, y = the price of the brandy } per dozen;

$$\text{then } 2x + y = 3 \times 78 = 234,$$
$$\text{and } 7x + 2y = 9 \times 79 = 711;$$

but the first equation being multiplied by 2, } $4x + 2y = 468$;

$$\therefore \text{by subtraction, } 3x = 243,$$
$$\text{and } x = 81;$$
$$\text{whence, } y = 234 - 2x = 234 - 162 = 72;$$

$\therefore$ the price of the sherry was 81*s*., and of brandy 72*s*.

19. A corn factor, mixes wheat-flour, which costs him 10 shillings a bushel, with barley-flour, which costs him 4 shillings a bushel, in such proportion, as to gain $43\frac{3}{4}$ *per cent.* by selling the mixture at 11 shillings a bushel. Required the proportion.

Let the proportion be $x : y$;

then $10x + 4y$ = the cost of $x + y$ bushels,

and $11x + 11y$ = the selling price;

$\therefore x + 7y$ = the gain;

whence,

$$10x+4y : x+7y :: 100 : 43\tfrac{3}{4} :: 400 : 175 :: 16 : 7,$$

$$\text{and } 5x+2y : x+7y :: 8 : 7;$$

$$\therefore 35x+14y=8x+56y;$$

$$\text{by transposition, } 27x=42y,$$

$$\text{and } 9x=14y;$$

$$\therefore x:y :: 14:9,$$

and $\therefore$ he must mix 14 bushels of wheat-flour with 9 of barley.

---

20. A number consisting of 2 digits when divided by 4, gives a certain quotient and a remainder of 3; when divided by 9 gives another quotient and a remainder of 8. Now the *value* of the digit on the left hand is equal the quotient which was got when the number was divided by 9; and the other digit is equal $\frac{1}{17}$th of the quotient got when the number was divided by 4. Required the number.

$$\text{Let } x \text{ and } y = \text{the digits in order;}$$

$$\text{then } 10x+y = \text{the number,}$$

$$\text{and } \frac{10x+y}{9} = x + \frac{8}{9};$$

$$\therefore 10x+y = 9x+8;$$

$$\text{by transposition, } x+y=8;$$

$$\text{also } \frac{10x+y}{4} = \frac{3}{4} + 17y;$$

$$\therefore 10x+y = 3+68y;$$

$$\text{by transposition, } 10x-67y=3;$$

$$\text{but from the preceding equation } 10x+10y=80;$$

$$\therefore \text{by subtraction, } 77y=77,$$

$$\text{and } y=1;$$

$$\therefore x = 8-y = 7,$$

and the number required is 71.

21. A man and his wife could drink a barrel of beer in 15 days After drinking together 6 days, the woman alone drank the remainder in 30 days. In what time would either alone drink a barrel?

Let x = the number of days in which the man could drink it,
and y = the number in which the woman could drink it;

$$\text{then } \frac{15}{x} + \frac{15}{y} = 1,$$

$$\text{and } \frac{6}{x} + \frac{6}{y} + \frac{30}{y} = 1,$$

$$\text{or } \frac{6}{x} + \frac{36}{y} = 1;$$

hence from the first equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{15}$,

$$\text{from the last } \frac{1}{x} + \frac{6}{y} = \frac{1}{6};$$

$$\therefore \text{ by subtraction, } \frac{5}{y} = \frac{1}{6} - \frac{1}{15} = \frac{3}{30} = \frac{1}{10},$$

$$\text{and } \therefore y = 50;$$

$$\text{also } \frac{1}{x} = \frac{1}{15} - \frac{1}{y} = \frac{1}{15} - \frac{1}{50} = \frac{7}{150};$$

$$\text{and } \therefore x = 21\tfrac{3}{7};$$

$\therefore$ the man would drink it in $21\frac{3}{7}$ days, and the woman in 50 days.

---

22. A purse holds 19 crowns and 6 guineas. Now 4 crowns and 5 guineas fill $\frac{17}{63}$ of it. How many will it hold of each?

Let x = the number of crowns,
and y = the number of guineas;

then $x : 1 :: 4$: the space occupied by 4 crowns $= \frac{4}{x}$.

In the same way, the space occupied by 5 guineas $= \frac{5}{y}$;

$$\therefore \frac{4}{x} + \frac{5}{y} = \frac{17}{63},$$

$$\text{and } \frac{19}{x} + \frac{6}{y} = 1.$$

The first equation being multiplied by 6, and the second by 5,

$$\frac{24}{x} + \frac{30}{y} = \frac{34}{21},$$

$$\text{and } \frac{95}{x} + \frac{30}{y} = 5;$$

$$\therefore \text{by subtraction, } \frac{71}{x} = 5 - \frac{34}{21} = \frac{71}{21};$$

$$\therefore x = 21,$$

$$\text{and } \frac{5}{y} = \frac{17}{63} - \frac{4}{21} = \frac{5}{63};$$

$$\therefore y = 63;$$

$\therefore$ the purse would hold 21 crowns, or 63 guineas.

23. Some smugglers discovered a cave, which would exactly hold the cargo of their boat, viz. 13 bales of cotton, and 33 casks of rum. Whilst they were unloading, a custom-house cutter coming in sight, they sailed away with 9 casks and 5 bales, leaving the cave two-thirds full. How many bales or casks would it hold?

Let x = the number of bales,
and y = the number of casks;

$$\text{then } \frac{13}{x} + \frac{33}{y} = 1,$$

$$\text{and } \frac{5}{x} + \frac{9}{y} = \frac{1}{3}.$$

Multiplying the first equation by 3, and the second by 11,

$$\text{then } \frac{39}{x} + \frac{99}{y} = 3,$$

$$\text{and } \frac{55}{x} + \frac{99}{y} = \frac{11}{3};$$

$$\therefore \text{ by subtraction, } \frac{16}{x} = \frac{2}{3};$$

$$\therefore 2x = 48,$$

$$\text{and } x = 24;$$

$$\text{consequently } \frac{9}{y} = \frac{1}{3} - \frac{5}{x} = \frac{1}{3} - \frac{5}{24} = \frac{3}{24} = \frac{1}{8},$$

$$\text{and } y = 72;$$

and ∴ the cave would hold 24 bales, or 72 casks.

24. Round two wheels, whose circumferences are as 5 to 3, two ropes are wrapped, whose difference exceeds the difference of the circumferences by 280 yards. Now the larger rope applied to the larger wheel wraps round it a certain number of times, greater by 12 than the smaller round the smaller wheel; and if the larger wheel turns round 3 times as quick as the other, the ropes will be discharged at the same time. Required the lengths of the ropes and the circumferences of the wheels.

Let $5x$ and $3x$ = the circumferences of the wheels (in yards);

then $2x + 280$ = the difference of the ropes,

and ($15x : 3x$ ::) $5 : 1$ :: the length of the longer string : the length of the shorter.

Let ∴ $5y$ and y = the lengths of the ropes (in yards);

$$\text{then } 4y = 2x + 280,$$

$$\text{and } \frac{5y}{5x} = \frac{y}{3x} + 12;$$

$$\text{or by transposition, } \frac{2y}{3x} = 12;$$

$$\therefore 4y = 72x,$$

$$\text{and } \therefore 72x = 2x + 280;$$

$$\text{by transposition, } 70x = 280,$$

$$\text{and } x = 4;$$

$$\therefore y = 72;$$

$\therefore$ the circumferences of the wheels are 20 and 12 yards, and the lengths of the strings 360 and 72 yards.

25. Three guineas were to be raised on two estates, to be charged proportionably to their values. Of this sum, *A*'s estate which was 4 acres more than *B*'s, but worse by 2 shillings an acre, paid £.1. 15*s*. But had *A* possessed 6 acres more, and *B*'s land been worth 3 shillings an acre less, it would have paid £.2. 5*s*. Required the values of the estates.

Let x = the number of acres *B* had;
then $x + 4$ = the number *A* had.
Let y = the value of an acre of *A*'s land;
$\therefore y + 2$ = the value of an acre of *B*'s,

$$\text{and } (x+4) . y : x . (y+2) :: 35 : 28 :: 5 : 4;$$

$$\therefore 4xy + 16y = 5xy + 10x,$$

$$\text{and } xy = 16y - 10x.$$

$$\text{Again, } (x+10) . y : x . (y-1) :: 45 : 18 :: 5 : 2;$$

$$\therefore 2xy + 20y = 5xy - 5x,$$

$$\text{and by transposition, } 3xy = 20y + 5x,$$

$$\text{whence } 48y - 30x = 20y + 5x;$$

$$\text{by transposition, } 28y = 35y,$$

$$\text{and } 4y = 5x,$$

which value substituted in the first equation, gives

$$xy = 20x - 10x = 10x;$$

$$\text{and } \therefore y = 10,$$

$$\text{whence } x = \frac{4y}{5} = 8;$$

$\therefore$ the value of *A*'s estate $= (x+4).y = 120 = £.6$;
and the value of *B*'s $= x.(y+2) = 96 = £.4.\ 16s.$

---

26. A coach set out from Cambridge to London with a certain number of passengers, 4 more being on the outside than within. Seven outside passengers could travel at 2 shillings less expence than 4 inside. The fare of the whole amounted to £.9. But at the end of half the journey, it took up 3 more outside and one more inside passengers; in consequence of which the fare of the whole became increased in the proportion of 17 to 15. Required the number of passengers, and the fare of the inside and outside.

Let $x =$ the number of inside } passengers,
$\therefore x + 4 =$ the number of outside }
and $y =$ the fare of an outside passenger;

$$\therefore \frac{7y+2}{4} = \text{the fare of an inside passenger,}$$

$$\text{and } \frac{7xy+2x}{4} + y.(x+4) = 180.$$

Also $\frac{3y}{2} + \frac{7y+2}{8} =$ the fare of the passengers taken up half way $= \frac{19y+2}{8}$;

$$\therefore \frac{19y+2}{8} : 180 :: 2 : 15,$$

$$\text{or } \frac{19y+2}{8} : 12 :: 2 : 1;$$

$$\therefore \frac{19y+2}{8} = 24,$$

$$\text{and } 19y + 2 = 192;$$

$\therefore$ by transposition, $19y = 190$,
and $y = 10$;

$\therefore$ from the first equation, $\frac{70x+2x}{4} + 10.(x+4) = 180$;

by transposition, $28x = 140$,
and $x = 5$;

$\therefore$ there were 5 inside, and 9 outside passengers, and the fares were 18 and 10 shillings, respectively.

---

27. In one of the corners of a rectangular garden there is a fish-pond, whose area is one-ninth part of the whole garden; the periphery of the garden exceeding that of the fish-pond by 200 yards. Also if the greater side be increased by 3 yards, and the other by 5 yards, the garden will be enlarged by 645 square yards. The fish-pond is a rectangle about the same diameter with the garden. Required the periphery of the garden, and the length of each side.

Let x = the length of the lesser side,
and y = the length of the greater;

$\therefore \frac{x}{3}$ and $\frac{y}{3}$ = the lengths of the lesser and greater sides of the fish-pond, (*Eucl.* B. 6. Prop. 24.)

Also $2.(x+y)$ = the periphery of the garden;

and $\frac{2.(x+y)}{3}$ = the periphery of the fish-pond;

$$\therefore 2.(x+y) - \frac{2}{3}.(x+y) = 200,$$

$$\text{or } \frac{2}{3}.(x+y) = 100;$$

$$\therefore x+y = 150.$$

Also $(y+3).(x+5) = xy + 645$;
$\therefore$ by transposition, $3x + 5y = 630$,

but from the former equation, $3x + 3y = 450$;

$$\therefore \text{by subtraction, } 2y = 180,$$
$$\text{and } y = 90;$$
$$\therefore x = 150 - y = 60;$$

and $\therefore$ the periphery $= 300$ yards, and the sides are 60 and 90 yards.

---

28. *A* and *B* each bought £.300 into the stocks, *A* into the *three per cents.*, and *B* into the *four's*. These stocks were at such a price that *B* received one pound interest more than *A*. When afterwards each of the stocks rose 10 *per cent.*, they sold out their money, and *A* found himself £.10 richer than *B*. Required the prices of the stocks.

Let x = the price of the three per cents.,
and y = the price of the four's;

$$\therefore x : 300 :: 3 : A\text{'s interest} = \frac{900}{x},$$

$$\text{and } y : 300 :: 4 : B\text{'s interest} = \frac{1200}{y};$$

$$\therefore \frac{900}{x} + 1 = \frac{1200}{y}.$$

Again, $x : x + 10 :: 300$: what *A* received when selling $= \dfrac{300.(x + 10)}{x}$;

and in the same way *B* received $\dfrac{300.(y + 10)}{y}$;

$$\therefore \frac{300.(x + 10)}{x} = \frac{300.(y + 10)}{y} + 10,$$

$$\text{and } 30 + \frac{300}{x} = 30 + \frac{300}{y} + 1;$$

$$\therefore \frac{300}{x} = \frac{300}{y} + 1,$$

and $\frac{900}{y} + 4 = \left(\frac{900}{x} + 1 =\right) \frac{1200}{y}$ from the first equation;

$$\therefore \text{ by transposition, } 4 = \frac{300}{y},$$

$$\text{and } y = 75;$$

$$\therefore \frac{300}{x} = 5,$$

$$\text{and consequently, } x = 60;$$

$\therefore$ the prices of the stocks were 60 and 75, *per* cent.

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29. £.500 was to be lent out at simple interest in two separate sums, the smaller at 2 *per* cent. more than the other. The interest of the greater sum was afterwards increased, and that of the smaller sum diminished by 1 *per* cent. By this, the interest of the whole was augmented by one-fourth of the former value. But if the interest of the greater sum had been so increased, without any diminution of the less, the interest of the whole would have been increased one-third. What were the sums and the rate *per* cent. of each?

$$\text{Let } x = \text{the less sum};$$

$$\therefore 500 - x = \text{the greater};$$

$$\text{let } y + 1 = \text{the interest of the less};$$

$$\therefore y - 1 = \text{the interest of the greater},$$

and $\frac{x \cdot (y+1)}{100} + \frac{(500-x) \cdot (y-1)}{100} = \frac{x}{50} + 5y - 5 =$ the former interest,

and $\frac{xy}{100} + \frac{(500-x) \cdot y}{100} = 5y =$ the second interest;

$$\therefore 5y = \frac{5}{4} \cdot \left(\frac{x}{50} + 5y - 5,\right)$$

$$\text{and } \therefore 4y = \frac{x}{50} + 5y - 5;$$

$$\text{by transposition, } y = 5 - \frac{x}{50}.$$

Again, the third interest $= \frac{x \cdot (y+1)}{100} + \frac{(500-x) \cdot y}{100} =$

$$\frac{x}{100} + 5y;$$

$$\therefore \left(\frac{x}{100} + 5y\right) \cdot 3 = 4 \cdot \left(\frac{x}{50} + 5y - 5,\right)$$

$$\text{or } \frac{3x}{100} + 15y = \frac{8x}{100} + 20y - 20;$$

by transposition, $\left(20 - \frac{5x}{100} =\right) 20 - \frac{x}{20} = 5y = 25 - \frac{x}{10}$ from the former equation;

$$\therefore \frac{x}{20} = 5,$$

$$\text{and } x = 100;$$

$$\text{also } y + 1 = 6 - \frac{x}{50} = 4;$$

$\therefore$ the sums were 100 and 400 pounds,
and the rates of interest £.4 and £.2, respectively.

Sect. VIII.

Examples of the Solution of Problems producing pure Equations.

1. What two numbers are those, whose sum is to the greater as 10 to 7; and whose sum multiplied by the less produces 270?

$$\text{Let } 10x = \text{their sum};$$
$$\therefore 7x = \text{the greater number},$$
$$\text{and } 3x = \text{the less};$$
$$\text{whence } 30x^2 = 270,$$
$$\text{and } x^2 = 9;$$
$$\therefore x = \pm 3,$$
and the numbers are ± 21 and ± 9.

2. There are two numbers in the proportion of 4 to 5, the difference of whose squares is 81. What are those numbers?

$$\text{Let } 4x \text{ and } 5x = \text{the numbers};$$
$$\text{then } (25x^2 - 16x^2 =)\ 9x^2 = 81;$$
$$\therefore x^2 = 9,$$
$$\text{and } x = \pm 3,$$
and the numbers are ± 12 and ± 15.

3. What two numbers are those, whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?

$$\text{Let } 2x = \text{their difference;}$$
$$\therefore\ 9x = \text{the greater,}$$
$$\text{and } 7x = \text{the less;}$$
$$\therefore\ (81x^2 - 49x^2 =)\ 32x^2 = 128,$$
$$\text{and } x^2 = 4;$$
$$\therefore\ x = \pm 2,$$
and the numbers are $\pm$ 18 and $\pm$ 14.

4. A mercer bought a piece of silk for £.16. 4*s.*; and the number of shillings which he paid for a yard was to the number of yards as 4 : 9. How many yards did he buy, and what was the price of a yard?

$$\text{Let } 4x = \text{the number of shillings he paid for a yard;}$$
$$\therefore\ 9x = \text{the number of yards,}$$
$$\text{and } 36x^2 = (\text{the price of the whole} =)\ 324;$$
$$\therefore\ x^2 = 9,$$
$$\text{and } \therefore\ x = \pm 3;$$
consequently there were 27 yards, at 12*s.* per yard.

4. It is required to divide the number 18 into two such parts, that the squares of those parts may be in the proportion of 25 to 16.

$$\text{Let } x = \text{the greater part;}$$
$$\text{then } 18 - x = \text{the less;}$$
$$\therefore\ x^2 : \overline{18 - x}|^2 :: 25 : 16,$$
and (*Alg.* 188.) $x : 18 - x :: 5 : 4$;
$$\therefore\ x : 18 :: 5 : 9,$$
$$\text{and } x : 2 :: 5 : 1;$$
$$\therefore\ x = 10,$$
and the parts are 10 and 8.

5. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$; the sum of whose squares is 724.

Reducing the fractions to a common denominator, the required numbers will evidently be in the proportion of 6, 8, and 9;

let $\therefore\ 6x$, $8x$, and $9x$, represent the numbers;

then $(36x^2 + 64x^2 + 81x^2 =)\ 181x^2 = 724$;

$$\therefore\ x^2 = 4,$$

$$\text{and } x = \pm 2,$$

and consequently, the numbers are $\pm$ 12, $\pm$ 16, and $\pm$ 18.

---

6. It is required to divide the number 14 into two such parts that the quotient of the greater part divided by the less, may be to the quotient of the less divided by the greater as 16 : 9.

$$\text{Let } x = \text{the greater part;}$$

$$\therefore\ 14 - x = \text{the less,}$$

$$\text{and } \frac{x}{14 - x} : \frac{14 - x}{x} :: 16 : 9;$$

$$\text{or } x^2 : \overline{14 - x}|^2 :: 16 : 9;$$

$$\therefore\ (\textit{Alg. } 188.)\ x : 14 - x :: 4 : 3,$$

$$\text{and } x : 14 :: 4 : 7;$$

$$\therefore\ x : 2 :: 4 : 1,$$

$$\text{and } x = 8;$$

$\therefore$ the parts are 8 and 6.

---

7 What two numbers are those whose difference is to the less as 4 to 3; and their product multiplied by the less is equal to 504?

$$\text{Let } 4x = \text{the difference;}$$

$$\text{then } 3x = \text{the less,}$$

$$\text{and } 7x = \text{the greater};$$
$$\text{whence } 63x^3 = 504,$$
$$\text{or } x^3 = 8;$$
$$\therefore x = 2;$$

and the numbers are 14 and 6.

---

8. What two numbers are as 5 to 4, the sum of whose cubes is 5103?

$$\text{Let } 5x \text{ and } 4x = \text{the numbers};$$
$$\therefore (125x^3 + 64x^3 =)\ 189x^3 = 5103,$$
$$\text{and } x^3 = 27;$$
$$\therefore x = 3,$$

and the numbers are 15 and 12.

---

9. A number of boys set out to rob an orchard, each carrying as many bags as there were boys in all, and each bag capable of containing 4 times as many apples as there were boys. They filled their bags, and found the number of apples was 2916. How many boys were there?

$$\text{Let } x = \text{the number of boys};$$
$$\text{then } x^2 = \text{the number of bags},$$
$$\text{and } 4x^3 = \text{the number of apples};$$
$$\therefore 4x^3 = 2916,$$
$$\text{and } x^3 = 729;$$
$$\therefore x = 9;$$

$\therefore$ there were 9 boys.

---

10. A person bought for one crown as many pounds of sugar, as were equal to half the number of crowns he laid out. In selling the sugar he received for every 25lbs. as many

crowns as the whole had cost him; and he received on the whole 20 crowns. How many crowns did he lay out, and what did he give for a pound?

Let x = the number of crowns he laid out;

$\therefore \frac{x}{2}$ = the number of lbs. for one crown,

and $\frac{x^2}{2}$ = the number of lbs. in all,

and $\frac{x}{25}$ = the selling price of one lb.;

$$\therefore \frac{x^2}{2} \times \frac{x}{25} = 20,$$

$$\text{and } x^3 = 1000;$$

$$\text{whence } x = 10;$$

$\therefore$ he laid out 10 crowns, and gave one shilling for a lb.

---

11. A detachment from an army was marching in regular column with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in five lines. Required the number of men.

Let x = the number in front;

$\therefore x + 5$ = the number in depth,

$$\text{and } x^2 + 5x = 5x + 4225,$$

$$\therefore x^2 = 4225,$$

$$\text{and } x = \pm 65;$$

$\therefore$ the number of men $= 5x + 4225 = 4550$, the negative value not answering the conditions of the problem.

---

12. A number of shillings were placed at equal distances on a table so as to form the sides of an equilateral triangle; then from the middle of each side a number of shillings, equal to

the square root of the number in the side were taken, and placed upon the corner shilling opposite to that side; it then appeared that the number on each side was to the number previously upon it, as 5 to 4. Required the number of shillings on one side at first.

$$\text{Let } x^2 = \text{the number};$$
$$\text{then } x^2 + x : x^2 :: 5 : 4,$$
$$\text{and } x : x^2 :: 1 : 4;$$
$$\therefore x^2 = 4x,$$
$$\text{and } x = 4;$$
$$\text{whence } x^2 = 16 = \text{the number required.}$$

13. A certain sum of money is divided every week among the resident members of a corporation. It happened one week that the number resident was the square root of the number of pounds to be divided. Two men however, coming into residence the week after, diminished the dividend of each of the former individuals £.1. 6*s*. 8*d*, What was the sum to be divided?

Let x^2 = the number of pounds;
then x = the number of men resident, and also = the sum each received.

$$\text{Hence, } x - \frac{4}{3} = \frac{x^2}{x+2};$$
$$\text{or } x^2 + \frac{2}{3}x - \frac{8}{3} = x^2;$$
$$\text{by transposition, } \frac{2}{3}x = \frac{8}{3},$$
$$\text{and } x = 4;$$
$$\therefore x^2 = 16 = \text{the sum required.}$$

14. Two partners *A* and *B* dividing their gain (£.60.), *B* took £.20; *A*'s money continued in trade 4 months, and if the

number 50 be divided by *A*'s money, the quotient will give the number of months that *B*'s money, which was £.100, continued in trade. What was *A*'s money, and how long did *B*'s money continue in trade?

Suppose *A*'s money was x pounds;

$\therefore \frac{50}{x}$ = the number of months *B*'s money was in trade,

and since *B* gained £.20, *A* gained £.40;

$$\therefore 4x : \frac{5000}{x} :: 2 : 1;$$

$$\text{or } x : \frac{2500}{x} :: 1 : 1;$$

$$\therefore x^2 = 2500,$$

$$\text{and } x = \pm 50;$$

$\therefore$ *A*'s money was £.50, and *B*'s money was one month in trade.

---

15. Two workmen *A* and *B* were engaged to work for a certain number of days at different rates. At the end of the time, *A* who had played 4 of those days, had 75 shillings to receive; but *B* who played 7 of those days, received only 48 shillings. Now had *B* only played 4 days, and *A* played 7 days, they would have received exactly alike. For how many days were they engaged; how many did each work, and what had each *per* day?

Let x = the number of days for which they were engaged;

$\therefore x - 4$ = the number *A* worked,

and $x - 7$ = the number *B* worked,

and $\frac{75}{x-4}$ = the number of shillings *A* received *per* day,

and $\frac{48}{x-7}$ = the number of shillings *B* received *per* day;

$$\therefore \frac{75 \cdot (x-7)}{x-4} = \frac{48 \cdot (x-4)}{x-7},$$

$$\text{and } 25 \cdot \overline{x-7}|^2 = 16 \cdot \overline{x-4}|^2;$$

$$\therefore 5 \cdot (x-7) = \pm 4 \cdot (x-4);$$

$$\therefore x = 19, \text{ or } \frac{17}{3};$$

and ∴ they were engaged to work 19 days,
and *A* worked 15, and *B* 12 days,
and *A* received 5 shillings, and *B* 4 shillings *per* day.

---

16. Two travellers *A* and *B* set out to meet each other, *A* leaving the town *C* at the same time that *B* left *D*. They travelled the direct road *CD*, and on meeting it appeared that *A* had travelled 18 miles more than *B*; and that *A* could have gone *B*'s journey in $15\frac{3}{4}$ days, but *B* would have been 28 days in performing *A*'s journey. What was the distance between *C* and *D*.

Let $x =$ the number of miles *A* has travelled;
$\therefore x - 18 =$ the number *B* has travelled,
and $x - 18 : x :: 15\frac{3}{4}$: the number of days *A* travelled
$= \frac{63x}{4 \cdot (x-18)}$;
also $x : x - 18 :: 28$: the number of days *B* travelled
$= \frac{28 \cdot (x-18)}{x}$;

$$\therefore \frac{28 \cdot (x-18)}{x} = \frac{63x}{4 \cdot (x-18)};$$

$$\text{or } 16 \cdot \overline{x-18}|^2 = 9x^2;$$

$$\therefore 4 \cdot (x-18) = \pm 3x,$$

$$\text{and } x = 72, \text{ or } 10\tfrac{2}{7};$$

whence *A* travelled 72, and *B* 54 miles;
and ∴ the whole distance *CD* 126 miles.

17. *A* and *B* lay out some money on speculation. *A* disposes of his bargain for £.11, and gains as much *per* cent. as *B* lays out; *B*'s gain is £.36, and it appears that *A* gains four times as much *per* cent. as *B*. Required the capital of each.

Let $4x = $ *B*'s capital, and ∴ *A*'s gain *per* cent.;

then $x = $ *B*'s gain *per* cent.,

and $100 : 4x :: x : 36$;

$$\therefore 4x^2 = 36 \times 100,$$

$$\text{and } x^2 = 9 \times 100;$$

$$\therefore x = \pm 30,$$

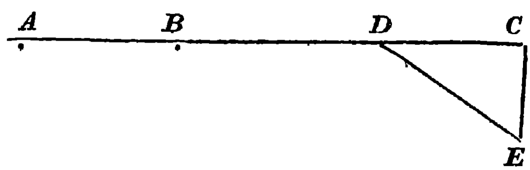
and ∴ *B*'s capital $= 120$,

and $220 : 100 :: 11 :$ *A*'s capital $= \dfrac{11 \times 10}{22} = 5.$

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18. The Captain of a privateer descrying a trading vessel 7 miles ahead, sailed 20 miles in direct pursuit of her, and then observing the trader steering in a direction perpendicular to her former course, changed his own course so as to overtake her without making another tack. On comparing their reckonings it was found, that the privateer had run at the rate of 10 knots in an hour, and the trading vessel at the rate of 8 knots in the same time. Required the distance sailed by the privateer.

Let *A*, *B*, be the original places of the privateer and trader, *E* the point of concourse, *D* the place where



the captain changed his course, *CE* being perpendicular to *AC*. $AB = 7$, $AD = 20$.

Now $(10 : 8 ::)\ 5 : 4$ :: the velocity of the privateer : the velocity of the trader

$:: AD : BC :: 20 : BC$;

$$\therefore BC = \frac{20 \times 4}{5} = 16;$$

$$\therefore DC = 16 - BD = 16 - 13 = 3,$$

and $DE : CE :: 5 : 4$. Let $CE = x$;

$$\therefore \sqrt{9 + x^2} : x :: 5 : 4,$$

and $9 + x^2 : x^2 :: 25 : 16$;

$$\therefore 9 : x^2 :: 9 : 16;$$

$$\therefore x^2 = 16, \text{ and } x = \pm 4,$$

and $\therefore DE = 5$, and $AD + DE = 25$.

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19. A vintner draws a certain quantity of wine out of a full vessel that holds 256 gallons; and then filling the vessel with water, draws off the same quantity of liquor as before, and so on, for four draughts, when there were only 81 gallons of pure wine left. How much wine did he draw each time?

Let $x =$ the number of gallons drawn the first time;
$\therefore 256 - x =$ the quantity of wine left,
and $256 : 256 - x :: x :$ the quantity of wine drawn

the second time $= \frac{x \cdot (256 - x)}{256}$;

$$\therefore 256 - x - \frac{x \cdot (256 - x)}{256} = \frac{\overline{256 - x}|^2}{256} = \text{the}$$

quantity left after the second draught.

In the same way, $\frac{\overline{256 - x}|^2}{256} \cdot x =$ the quantity drawn the third time,

and $\frac{\overline{256 - x}|^3}{\overline{256}|^2} =$ the quantity left,

and $\overline{\frac{256-x}{256}}\Big|^3 . x$ and $\frac{\overline{256-x}|^4}{\overline{256}|^3}$ = the quantities drawn and left the fourth time;

$$\text{whence } \frac{\overline{256-x}|^4}{\overline{256}|^3} = 81,$$

$$\text{and } 256 - x = \overline{256}|^{\frac{3}{4}} \times 3 = 64 \times 3 = 192;$$

$$\therefore \text{ by transposition, } 64 = x,$$

and the quantities drawn off each time were 64, 48, 36, and 27 gallons.

---

20. What two numbers are those, whose difference multiplied by the greater produces 40, and by the less 15?

Let x = the greater,
and y = the less;

$$\therefore x^2 - xy = 40,$$
$$\text{and } xy - y^2 = 15;$$

$$\therefore \text{ by subtraction, } x^2 - 2xy + y^2 = 25,$$
$$\text{and } x - y = \pm 5;$$
$$\therefore \text{ from the first equation } \pm 5x = 40,$$
$$\text{and } x = \pm 8,$$
$$\text{and from the second equation } \pm 5y = 15,$$
$$\text{and } y = \pm 3;$$
$$\therefore \text{ the numbers are } \pm 8, \text{ and } \pm 3.$$

---

21. What two numbers are those whose difference multiplied by the less produces 42, and by their sum 133?

Let x = the greater,
and y = the less;

$$\therefore (x - y) . y = 42,$$
$$\text{and } (x - y) . (x + y) = 133;$$

$\therefore$ by subtraction, $(x - y) \cdot x = 91$, subtracting the first equation from this, $(x - y) \cdot (x - y) = 49$;

$$\text{or } \overline{x - y}|^2 = 49;$$

$$\therefore x - y = \pm 7;$$

$$\text{whence } \pm 7y = 42,$$

$$\text{and } y = \pm 6,$$

$$\text{and } x = \pm 7 + y = \pm 13;$$

$\therefore$ the numbers are ± 13, and ± 6.

---

22. In a mixture of rum and brandy, the difference between the quantities of each is to the quantity of brandy as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum as 4 to the number of gallons of brandy. How many gallons are there of each?

Let $x =$ the number of gallons of rum,
and $y =$ the number of gallons of brandy;

$$\therefore x - y : y :: 100 : x,$$

$$\text{and } x : x - y :: y : 4;$$

$$\therefore \textit{ex æquali, } x : y :: 25y : x,$$

$$\text{and } x^2 = 25y^2;$$

$\therefore x = \pm 5y$, the negative value not answering the conditions of the problem.

Now from the second proportion $5y : 4y :: y : 4$;

$$\text{or } 5 : 1 :: y : 1;$$

$$\therefore y = 5,$$

$$\text{and } x = 25;$$

$\therefore$ there are 25 gallons of rum, and 5 of brandy.

---

23. What two numbers are those whose difference being multiplied by the greater, and the product divided by the less, quotes 24; but if their difference be multiplied by the less, and the product divided by the greater, the quotient is 6?

Let x = the greater,
and y = the less;

$$\text{then } (x - y) \cdot \frac{x}{y} = \pm 24,$$

$$\text{and } (x - y) \cdot \frac{y}{x} = 6;$$

dividing the first equation by the second, $\frac{x^2}{y^2} = 4$;

$$\therefore \frac{x}{y} = \pm 2;$$

$$\text{or } x = \pm 2y,$$

and $\therefore$ in the first case, $(x - y =)\ y = 12$, and $x = 24$;
but if $x = -2y$; $-3y \times -2 = 24$;
$\therefore y = 4$, and $x = -8$.

---

24. It is required to find two numbers such, that the product of the greater and square root of the less may be equal to 48, and the product of the less and square root of the greater may be 36.

Let x^2 and y^2 be the two numbers;
$\therefore x^2y = 48$, and $xy^2 = 36$;

$$\therefore \frac{48}{x} = (xy =)\ \frac{36}{y},$$

$$\text{or } \frac{4}{x} = \frac{3}{y};$$

$$\therefore y = \frac{3x}{4};$$

$$\text{whence } \frac{3x^3}{4} = 48,$$

and $x^3 = 64$;
$\therefore x = 4$,
and consequently, $y = 3$;
$\therefore$ the numbers are 16, and 9.

25. Find two numbers such that the square of the greater multiplied by the less may be equal to 448, and the square of the less multiplied by the greater may be 392.

Let $x =$ the greater and $y =$ the less;
then $x^2y = 448$, and $xy^2 = 392$;

$$\therefore \frac{448}{x} = (xy =) \frac{392}{y},$$

$$\text{or } \frac{8}{x} = \frac{7}{y};$$

$$\therefore x = \frac{8y}{7},$$

$$\text{and consequently, } \frac{8y^3}{7} = 392,$$

$$\text{or } \frac{y^3}{7} = 49;$$

$$\therefore y^3 = 343, \text{ and } y = 7;$$

$$\therefore x = 8;$$

$\therefore$ the numbers are 8, and 7.

---

26. *A* and *B* carried 100 eggs between them to market, and each received the same sum. If *A* had carried as many as *B* he would have received 18 pence for them, and if *B* had taken only as many as *A*, he would have received only 8 pence. How many had each?

Let $x =$ the number *A* had,
and $y =$ the number *B* had;

then $\frac{18}{y} =$ the price of one egg of *A*'s (in pence),

and $\frac{8}{x} =$ the price of one of *B*'s;

$$\therefore \frac{18x}{y} = \frac{8y}{x},$$

$$\text{and } 9x^2 = 4y^2;$$

$\therefore 3x = \pm 2y$, the negative value of which will not answer the conditions of the problem.

$$\text{Now } (x + y =) \; x + \frac{3x}{2} = 100;$$

$$\therefore (2x + 3x =) \; 5x = 200,$$

$$\text{and } x = 40;$$

$$\text{and } \therefore y = 60.$$

27. What two numbers are those, which being both multiplied by 27 the first product is a square, and the second the root of that square: but being both multiplied by 3, the first product is a cube, and the second the root of that cube.

Let x and y be the numbers;

$$\text{then } \sqrt{27x} = 27y,$$

$$\text{and } \therefore x = 27y^2;$$

$$\text{also } \sqrt[3]{3x} = 3y,$$

$$\text{and } \therefore x = 9y^3;$$

$$\text{whence } 9y^3 = 27y^2,$$

$$\text{and } y = 3;$$

$$\therefore x = 27y^2 = 243;$$

$\therefore$ the numbers are 243, and 3.

28. It is required to find the three sides of a right-angled triangle from the following data. The number of square feet in the area is equal to the number of feet in the hypothenuse + the sum in the other two sides; and the square described upon the hypothenuse is less than the square described upon a line equal in length to the two sides, by half the product of numbers representing the base and area.

Let x = the number of feet in the altitude,
and y = the number in the base;

$\therefore \sqrt{x^2+y^2}$ = the number in the hypothenuse, (*Eucl.* B. 1. p. 47.)

$$\text{and } \frac{xy}{2} = \text{the area};$$

$$\therefore \tfrac{1}{2}xy = \sqrt{x^2+y^2} + x + y;$$

$$\text{also } x^2+y^2 = (\overline{x+y})^2 - \tfrac{1}{4}xy^2 =)\, x^2+2xy+y^2 - \tfrac{1}{4}xy^2;$$

$$\therefore \text{ by transposition, } \tfrac{1}{4}xy^2 = 2xy,$$

$$\text{and } y = 8;$$

$$\text{hence from the first equation, } 4x = \sqrt{x^2+64} + x + 8,$$

$$\text{and by transposition, } 3x - 8 = \sqrt{x^2+64};$$

$$\therefore 9x^2 - 48x + 64 = x^2 + 64,$$

$$\text{and } 8x^2 = 48x;$$

$$\therefore x = 6;$$

$$\text{whence the hypothenuse } = \sqrt{64+36} = 10;$$

$\therefore$ the sides are 6, 8, and 10 feet, respectively.

---

29. A farmer has 2 cubical stacks of hay. The side of one is 3 yards longer than the side of the other; and the difference of their contents is 117 solid yards. Required the side of each.

Let x = the side of the greater,
and y = the side of the less;

$$\therefore x^3 - y^3 = 117,$$

$$\text{and } x - y = 3;$$

$$\text{cubing the latter equation, } x^3 - 3x^2y + 3xy^2 - y^3 = 27;$$

$$\text{but } x^3 \qquad - y^3 = 117;$$

$$\therefore \text{ by subtraction, } 3x^2y - 3xy^2 = 90,$$

$$\text{and } xy\,.\,(x-y) = 30,$$

$$\text{or } 3xy = 30;$$

$$\therefore xy = 10.$$

$$\text{Now } x^2 - 2xy + y^2 = 9,$$
$$\text{and } 4xy \quad = 40$$
$$\therefore \text{ by addition, } x^2 + 2xy + y^2 = 49,$$
$$\text{and } x + y = \pm 7;$$
$$\text{but } x - y = 3;$$
$$\therefore \text{ by addition, } 2x = 10, \text{ or } -4;$$
$$\therefore x = 5, \text{ or } -2,$$
$$\text{and by subtraction, } 2y = 4, \text{ or } -10;$$
$$\therefore y = 2, \text{ or } -5,$$

and the sides of the stacks are 5, and 2 yards, respectively.

30. When a parish was inclosed, the allotment of one of the proprietors consisted of two pieces of ground; one of which was in the form of a right-angled triangle; the other was a rectangle, one of the sides of which was equal to the hypothenuse of the triangle, the other, to half the greater side; but wishing to have his land in one piece, he exchanged his allotments for a square piece of ground of equal area, one side of which equalled the greater of the sides of the triangle which contained the right-angle. By this exchange he found that he had saved ten poles of railing. What are the respective areas of the triangle and rectangle; and what is the length of each of their sides?

Let $2x =$ the greater side of the triangle,

and $y =$ the less;

$\therefore \sqrt{4x^2+y^2} =$ the hypothenuse; and also the greater side of the rectangle,

and $x =$ the less side of the rectangle;

$\therefore xy =$ the area of the triangle,

and $x\sqrt{4x^2+y^2} =$ the area of the rectangle;

$$\therefore 4x^2 = xy + x\sqrt{4x^2+y^2},$$
$$\text{or } 4x - y = \sqrt{4x^2+y^2};$$

also $8x+10 = 2x+y+\sqrt{4x^2+y^2}+2x+2\sqrt{4x^2+y^2}$,

or $4x + 10 = y + 3\sqrt{4x^2 + y^2}$; in which equation substituting the value of $\sqrt{4x^2 + y^2}$ found above;

$$\therefore 4x + 10 = y + 3(4x - y) = 12x - 2y;$$

$$\therefore \text{by transposition, } 2y = 8x - 10,$$

$$\text{and } y = 4x - 5;$$

$$\therefore \text{from the first equation, } 5 = \sqrt{4x^2 + \overline{4x - 5}|^2},$$

$$\text{and } 25 = 4x^2 + 16x^2 - 40x + 25;$$

$$\text{by transposition, } 40x = 20x^2;$$

$$\therefore 2 = x,$$

$$\text{and } y = 4x - 5 = 3;$$

$\therefore$ the sides of the triangle are 3, 4, and 5; the sides of the rectangle are 2, and 5; and the areas of the triangle and rectangle are 6, and 10, respectively.

SECT. IX.

Examples of the Solution of Problems producing Adfected Quadratic Equations.

1. A MERCHANT sold a quantity of brandy for £.39, and gained as much *per* cent. as the brandy cost him. What was the price of the brandy?

Let x = the price of the brandy;

then $100 : x :: x :$ the gain $= \frac{x^2}{100}$,

and $\therefore \frac{x^2}{100} = 39 - x$,

or $x^2 = 3900 - 100x$;

by transposition, $x^2 + 100x = 3900$,

completing the square, $x^2 + 100x + \overline{50}|^2 = 3900 + 2500$
$= 6400$;

extracting the root, $x + 50 = \pm 80$;

$\therefore x = 30$, or -130;

$\therefore$ the price was £.30.

---

2. There are two numbers whose difference is 9, and their sum multiplied by the greater produces 266. What are those numbers?

Let x = the greater;

$\therefore x - 9 =$ the less,

and $x . (2x - 9) = 266$;

$\therefore x^2 - \frac{9}{2} . x = \frac{266}{2}$;

completing the square, $x^2 - \frac{9}{2}x + \frac{81}{16} = \frac{266}{2} + \frac{81}{16} = \frac{2209}{16}$;

extracting the root, $x - \frac{9}{4} = \pm \frac{47}{4}$;

$\therefore\ x = 14$, or $-\frac{19}{2}$;

$\therefore\ x - 9 = 5$, or $-\frac{37}{2}$,

and both values answer the conditions of the problem.

---

3. It is required to find two numbers, the first of which may be to the second as the second is to 16; and the sum of the squares of the numbers may be equal to 225.

Let x = the first number;

$\therefore\ \sqrt{16x}$ = the second,

and $x^2 + 16x = 225$;

completing the square, $x^2 + 16x + 64 = 225 + 64 = 289$;

extracting the root, $x + 8 = \pm 17$;

$\therefore\ x = 9$, or -25;

but as this latter value of x makes the second number an impossible quantity, 9 is the only value of x which answers the conditions, and therefore the numbers are 9 and 12.

---

4. Bought two sorts of linen for 6 crowns. An ell of the finer cost as many shillings as there were ells of the finer. Also 28 ells of the coarser (which was the whole quantity) were at such a price that 8 ells cost as many shillings as 1 ell of the finer. How many ells were there of the finer, and what was the value of each price?

Let x = the number of ells of the finer;

$\therefore\ x^2$ = the price of the finer (in shillings,)

and $8 : 28 :: x :$ the price of the coarser $= \frac{7x}{2}$;

$$\therefore\ x^2 + \frac{7x}{2} = 30;$$

completing the square, $x^2 + \frac{7}{2}x + \frac{49}{16} = 30 + \frac{49}{16} = \frac{529}{16}$;

extracting the root, $x + \frac{7}{4} = \pm\frac{23}{4}$,

and $x = 4$, or $-\frac{15}{2}$,

and $\therefore$ the price of the finer $= 16$ shillings, and of the coarser $= 14$ shillings.

---

5. Two partners *A* and *B* gained £.18 by trade. *A*'s money was in trade 12 months, and he received for his principal and gain £.26. Also *B*'s money, which was £.30, was in trade 16 months. What money did *A* put into trade?

Let $x =$ the number of pounds he put in;

$\therefore\ 26 - x =$ the number he gained,

and $12x + 16 \times 30 : 12x :: 18 : 26 - x$;

$\therefore\ x + 40 : x :: 18 : 26 - x$,

and $18x = 1040 - 14x - x^2$;

by transposition, $x^2 + 32x = 1040$;

completing the square, $x^2 + 32x + \overline{16}|^2 = 1040 + 256 = 1296$;

extracting the root, $x + 16 = \pm 36$;

$\therefore\ x = 20$, or -42,

and consequently *A* put £.20 into trade.

---

6. A person bought some sheep for £.72; and found that if he had bought 6 more for the same money, he would have paid £.1 less for each. How many did he buy, and what was the price of each?

Let x = the number of sheep bought;

then $\frac{72}{x}$ = the price of one (in pounds,)

and $\frac{72}{x+6}$ = the price of one, if he had bought 6 more;

$$\therefore \frac{72}{x+6} + 1 = \frac{72}{x};$$

$$\therefore 72x + x^2 + 6x = 72x + 432;$$

$$\therefore x^2 + 6x = 432;$$

completing the square, $x^2 + 6x + 9 = 441$;

extracting the root, $x + 3 = \pm 21$,

and $x = 18$, or -24,

and $\therefore$ he bought 18, and the price of one $= \frac{72}{18}$

$= 4$ pounds.

---

7. The plate of a looking-glass is 18 inches by 12, and is to be framed with a frame of equal width, whose area is to be equal to that of the glass. Required the width of the frame.

The area of the glass $= 12 \times 18 = 216$.

Let x = the width of the frame (in inches;)

then the area of the frame $= (18 + 2x) . (12 + 2x) - 216$,

and $\therefore (18 + 2x) . (12 + 2x) - 216 = 216$,

or $4x^2 + 60x = 216$,

and $x^2 + 15x = 54$;

completing the square, $x^2 + 15x + \overline{\frac{15}{2}}\Big|^2 = 54 + \frac{225}{4}$

$= \frac{441}{4}$;

extracting the root, $x + \frac{15}{2} = \pm \frac{21}{2}$;

$\therefore x = 3$, or -18,

and $\therefore$ the width must be 3 inches.

$$\text{extracting the root, } x + \frac{45}{2} = \pm \frac{195}{2},$$

$$\text{and } x = 75, \text{ or } - 120;$$

$$\text{and } \therefore \text{ the number bought was } 75,$$

$$\text{and the price} = \frac{4}{5}\text{£.} = 16 \text{ shillings.}$$

---

12. *A* and *B* set out from two towns which were at the distance of 247 miles, and travelled the direct road till they met. *A* went 9 miles a day; and the number of days, at the end of which they met, was greater by 3 than the number of miles which *B* went in a day. How many miles did each go?

Let $x =$ the number of days they travelled;

$\therefore 9x =$ the number of miles *A* went,

and $247 - 9x =$ the number *B* went,

and $\frac{247 - 9x}{x} =$ the number *B* went *per* day;

$$\therefore x - 3 = \frac{247 - 9x}{x},$$

$$\text{and } x^2 - 3x = 247 - 9x;$$

$$\text{by transposition, } x^2 + 6x = 247;$$

$$\text{completing the square, } x^2 + 6x + 9 = 256,$$

$$\text{extracting the root, } x + 3 = \pm 16,$$

$$\text{and } x = 13, \text{ or } - 19,$$

and $\therefore$ *A* went 117, and *B* 130 miles.

---

13. A person bought two pieces of cloth of different sorts; whereof the finer cost 4 shillings a yard more than the other; for the finer he paid £.18; but the coarser, which exceeded the finer in length by 2 yards, cost only £.16. How many yards were there in each piece, and what was the price of a yard of each?

Let x = the number of yards of the finer;

$\therefore x + 2$ = the number of yards of the coarser,

and $\frac{18}{x}$ = the price of a yard of the finer (in pounds);

also $\frac{16}{x+2}$ = the price of a yard of the coarser;

$$\therefore \frac{18}{x} = \frac{16}{x+2} + \frac{1}{5},$$

and $90x + 180 = 80x + x^2 + 2x$;

by transposition, $x^2 - 8x = 180$;

completing the square, $x^2 - 8x + 16 = 180 + 16 = 196$;

extracting the root, $x - 4 = \pm 14$;

$\therefore x = 18$, or -10,

consequently, there were 18 yards of the finer, and 20 of the coarser; and the prices were £.1, and 16 shillings, respectively.

---

14. *A* set out from *C* towards *D*, and travelled 7 miles a day. After he had gone 32 miles, *B* set out from *D* towards *C*, and went every day $\frac{1}{19}$th of the whole journey; and after he had travelled as many days as he went miles in one day, he met *A*. Required the distance of the places *C* and *D*.

Suppose the distance was x miles;

$\therefore \frac{x}{19}$ = the number of miles *B* travelled *per* day; and also = the number of days he travelled before he met *A*.

$$\therefore \frac{x^2}{361} + 32 + \frac{7x}{19} = x;$$

by transposition, $\frac{x^2}{361} - \frac{12x}{19} = -32$;

completing the square, $\frac{x^2}{361} - \frac{12x}{19} + 36 = 36 - 32 = 4$;

$$\text{extracting the root, } \frac{x}{19} - 6 = \pm 2;$$

$$\therefore \frac{x}{19} = 8, \text{ or } 4,$$

and $x = 152$, or 76, both which values answer the conditions of the problem. The distance therefore of C from D was 152, or 76 miles.

---

15. A and B sold 130 ells of silk, (of which 40 ells were A's, and 90 B's) for 42 crowns. Now A sold for a crown one-third of an ell more than B did. How many ells did each sell for a crown.

$$\left.\begin{array}{l}\text{Let } x = \text{the number } B \text{ sold} \\ \therefore x + \frac{1}{3} = \text{the number } A \text{ sold}\end{array}\right\} \text{for a crown,}$$

$$\text{and } x : 90 :: 1 : \text{the price of 90 ells} = \frac{90}{x},$$

$$\text{and } x + \frac{1}{3} : 40 :: 1 : \text{the price of 40 ells} = \frac{120}{3x+1};$$

$$\therefore 42 = \frac{90}{x} + \frac{120}{3x+1},$$

$$\text{or } 7 = \frac{15}{x} + \frac{20}{3x+1};$$

$$\text{whence, } 21x^2 + 7x = 45x + 15 + 20x;$$

$$\text{by transposition, } 21x^2 - 58x = 15,$$

$$\text{and } x^2 - \frac{58}{21}x = \frac{15}{21};$$

$$\text{completing the square, } x^2 - \frac{58}{21} \cdot x + \overline{\frac{29}{21}}\Big|^2 = \frac{15}{21} + \frac{841}{\overline{21}|^2} = \frac{1156}{\overline{21}|^2}$$

$$\text{extracting the root, } x - \frac{29}{21} = \pm \frac{34}{21};$$

$$\therefore x = 3, \text{ or } -\frac{5}{21};$$

whence, *B* sold 3 ells, and *A* $3\frac{1}{3}$, for a crown.

16. Three merchants *A*, *B*, and *C*, made a joint stock, by which they gained a sum less than that stock by £.80. *A*'s share of the gain was £.60; and his contribution to the stock was £.17 more than *B*'s Also *B* and *C* together contributed £.325. How much did each contribute?

Let x = the number of pounds that *A* contributed;
$\therefore x - 17$ = the number that *B* contributed,
and $325 - (x - 17) = 342 - x$ = the number that *C* contributed;

$\therefore 325 + x$ = the whole stock,

and $325 + x - 80 = 245 + x$ = the whole gain;

$$\therefore 325 + x : x :: 245 + x : 60,$$

$$\text{and } x^2 + 245x = 60x + 19500;$$

$$\text{by transposition, } x^2 + 185x = 19500;$$

completing the square, $x^2 + 185x + \left(\frac{185}{2}\right)^2 = 19500 + \frac{34225}{4} = \frac{112225}{4}$;

$$\text{extracting the root, } x + \frac{185}{2} = \pm\frac{335}{2},$$

$$\text{and } x = 75, \text{ or } -260;$$

$\therefore$ the stocks of *A*, *B* and *C* were 75, 58, and 267 pounds, respectively.

17. The joint stock of 2 partners *A* and *B* was £.416. *A*'s money was in trade 9 months, and *B*'s 6 months: when they shared stock and gain, *A* received £.228, and *B* £.252. What was each man's stock?

Let $x = A$'s stock;
$\therefore 228 - x =$ his gain;
also $416 - x = B$'s stock,
and $x - 164 =$ his gain;
and $\therefore 64 =$ the whole gain,
and $9x + 6.(416 - x) : 9x :: 64 : 228 - x$;
or $3x + 2.(416 - x) : 3x :: 64 : 228 - x$;
$\therefore 192x = (x + 832).(228 - x) = 189696 - 604x - x^2$;
by transposition, $x^2 + 796x = 189696$;
completing the square, $x^2 + 796x + \overline{398}|^2 = 189696 + 158404 = 348100$;
extracting the root, $x + 398 = \pm 590$,
and $x = 192$, or $- 988$;
$\therefore$ the stocks were £.192, and £.224.

---

18. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 to their number, a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square. Required the number of men in the hollow square.

Let $x =$ the number of men in a side of the hollow square;
$\therefore x^2 - \overline{x - 6}|^2 =$ the whole number of men,
and $x^2 - \overline{x-6}|^2 + 25 = \overline{x^{\frac{1}{2}} + 22}|^2$,
or $12x - 36 + 25 = x + 44x^{\frac{1}{2}} + 484$;
$\therefore$ by transposition, $11x - 44x^{\frac{1}{2}} = 495$;
or $x - 4x^{\frac{1}{2}} = 45$;
completing the square, $x - 4x^{\frac{1}{2}} + 4 = 49$;
extracting the root, $x^{\frac{1}{2}} - 2 = \pm 7$;
$\therefore x^{\frac{1}{2}} = 9$, or $- 5$,

and $x = 81$, or 25,

and $\therefore$ the whole number $= 936$.

---

19. A mercer bought a number of pieces of two different kinds of silk for £.92. 3*s.* There were as many pieces bought of each kind, and as many shillings paid *per* yard for them as a piece of that kind contained yards. Now 2 pieces, one of each kind, together measured 19 yards. How many yards were there in each?

Let $x =$ the number of yards in one piece; and $\therefore$ = the number of pieces, and also the number of shillings *per* yard;

$\therefore 19 - x =$ the number in the other,

and x^3, and $\overline{19 - x}|^3 =$ the whole prices of each kind;

$$\therefore x^3 + \overline{19 - x}|^3 = 1843,$$

$$\text{or } 57x^2 - 1083x + 6859 = 1843;$$

$$\text{by transposition, } 57x^2 - 1083x = -5016;$$

$$\text{or } x^2 - 19x = -88;$$

$$\text{completing the square, } x^2 - 19x + \overline{\frac{19}{2}}|^2 = \frac{361}{4} - 88 = \frac{9}{4};$$

$$\text{extracting the root, } x - \frac{19}{2} = \pm\frac{3}{2};$$

$$\therefore x = 11, \text{ or } 8;$$

$$\therefore 35 - x = 8, \text{ or } 11,$$

both which values answer the conditions of the problem;

$\therefore$ there were 11 yards in one, and 8 in the other.

---

20. A square court-yard has a rectangular gravel-walk round it. The side of the court wants 2 yards of being 6 times the breadth of the gravel-walk; and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 92. Required the area of the court.

Let x = the breadth of the walk (in yards,)

$\therefore\ 6x - 2$ = the side of the court,

and $4x - 2$ = the side of the interior square;

$\therefore\ \overline{6x-2}|^2 - \overline{4x-2}|^2$ = the area of the walk,

and $20x^2 - 8x - 92 = 4 \times (6x - 2;)$

by transposition, $20x^2 - 32x = 84$;

$$\therefore\ x^2 - \frac{8}{5}x = \frac{21}{5};$$

completing the square, $x^2 - \frac{8}{5}x + \frac{16}{25} = \frac{21}{5} + \frac{16}{25} = \frac{121}{25}$;

extracting the root, $x - \frac{4}{5} = \pm \frac{11}{5}$;

$$\therefore\ x = 3, \text{ or } -\frac{7}{5},$$

and $\overline{6x-2}|^2 = \overline{16}|^2 = 256$, the area required.

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21. A merchant bought 54 gallons of Coniac brandy, and a certain quantity of British. For the former he gave half as many shillings *per* gallon as there were gallons of British, and for the latter 4 shillings *per* gallon less. He sold the mixture at 10 shillings *per* gallon, and lost £.28. 16*s*. by his bargain. Required the price of the Coniac, and the number of gallons of British.

Let $2x$ = the number of gallons of British;

$\therefore\ x$ = the number of shillings one gallon of Coniac cost,

and $54x$ = the price of all the Coniac;

also $x - 4$ = the number of shillings one gallon of British cost,

and $2x^2 - 8x$ = the price of all the brandy;

$\therefore\ 2x^2 - 8x + 54x = 10\,.\,\overline{54 + 2x} + 576$;

by transposition, $2x^2 + 26x = 1116$,

or $x^2 + 13x = 558$;

completing the square, $x^2 + 13x + \overline{\frac{13}{2}}\Big|^2 = 558 + \frac{169}{4} = \frac{2401}{4}$;

extracting the root, $x + \frac{13}{2} = \pm \frac{49}{2}$,

and $x = 18$, or -31;

∴ he bought 36 gallons of British; the Coniac cost 18 shillings *per* gallon,

and ∴ the whole price = £.48. 12*s.*

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22. During the time that the shadow on a sun-dial, which shews true time, moves from one o'clock to five, a clock, which is too fast a certain number of hours and minutes, strikes a number of strokes = that number of hours and minutes, and it is observed that the number of minutes is less by 41 than the square of the number which the clock strikes at the last time of striking. The clock does not strike twelve during the time. How much is it too fast?

Let x = the number of hours too fast;

then the clock strikes, $(x + 2) + (x + 3) + (x + 4) + (x + 5)$ times $= 4x + 14$,

and the number of minutes $= x^2 + 10x + 25 - 41 = x^2 + 10x - 16$;

∴ $x + x^2 + 10x - 16 = 4x + 14$;

by transposition, $x^2 + 7x = 30$;

completing the square, $x^2 + 7x + \overline{\frac{7}{2}}\Big|^2 = 30 + \frac{49}{4} = \frac{169}{4}$;

extracting the root, $x + \frac{7}{2} = \pm \frac{13}{2}$;

∴ $x = 3$, or -10,

and the number of minutes $= 23$;

∴ the clock is too fast 3 hours and 23 minutes.

23. A vinter sold 7 dozen of sherry and 12 dozen of claret for £.50. He sold 3 dozen more of sherry for £.10 than he did of claret for £.6. Required the price of each.

Let x = the price of a dozen of sherry (in pounds;)

$\therefore\ x : 10 :: 1 :$ the number of dozens of sherry for £.10, $= \frac{10}{x}$,

and $\frac{10}{x} - 3 = \frac{10 - 3x}{x}$ = the number of dozens of claret for £.6;

$\therefore\ \frac{10 - 3x}{x} : 1 :: 6 :$ the price of a dozen of claret $= \frac{6x}{10 - 3x}$;

$$\therefore\ 7x + \frac{72x}{10 - 3x} = 50,$$

and $70x - 21x^2 + 72x = 500 - 150x$;

by transposition, $292x - 21x^2 = 500$,

$$\text{or } x^2 - \frac{292}{21} . x = - \frac{500}{21};$$

completing the square, $x^2 - \frac{292}{21}.x + \overline{\frac{146}{21}}\Big|^2 = \frac{21316}{\overline{21}|^2} - \frac{500}{21}$;

$$= \frac{10816}{\overline{21}|^2};$$

extracting the root, $x - \frac{146}{21} = \pm \frac{104}{21}$,

and $x = 2$, or $\frac{250}{21}$;

$\therefore$ the price of a dozen of sherry was £.2, and the price of a dozen of claret $= \frac{6x}{10 - 3x}$ = £.3.

24. *A* and *B* hired a pasture into which *A* put 4 horses, and *B* as many as cost him 18 shillings a week. Afterwards

B put in two additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Let x = the number of *B*'s horses at first;

then $\frac{18}{x}$ = the pay of each *per* week (in shillings;)

$\therefore \frac{72}{x}$ = what *A* paid,

and $\frac{72}{x} + 18$ = the price of the pasture;

also $x + 6$ = the whole number of horses in the second case;

$$\therefore x + 6 : x + 2 :: \frac{72}{x} + 18 : 20 :: 72 + 18x : 20x;$$

$$\therefore 20x^2 + 120x = 18x^2 + 108x + 144;$$

by transposition, $2x^2 + 12x = 144$,

or $x^2 + 6x = 72$;

completing the square, $x^2 + 6x + 9 = 72 + 9 = 81$;

extracting the root, $x + 3 = \pm 9$,

and $x = 6$, or -12;

$\therefore$ *B* had 6 horses in the pasture at first, and $\frac{72}{x} + 18$ = 30 shillings, *per* week, was the price of the pasture.

25. An upholsterer has two square carpets divided into square yards by the lines of the pattern. Now he observes, that if he subtracts from the number of squares in the smaller carpet, the number of yards in the side of the other, the square of the remainder will exceed the difference of the number of squares in the smaller carpet, and the number of yards in its side, by 88. Also the difference of the lengths of the sides of the carpets is 6 feet. Required the size of each carpet.

Let x = the number of yards in a side of the less;

$\therefore x + 2$ = the number in a side of the greater,

and $\overline{x^2 - x - 2}|^2 = x^2 - x + 88 = x^2 - x - 2 + 90$;
by transposition, $\overline{x^2 - x - 2}|^2 - (x^2 - x - 2) = 90$;
completing the square, $\overline{x^2 - x - 2}|^2 - (x^2 - x - 2) + \frac{1}{4}$
$= 90 + \frac{1}{4} = \frac{361}{4}$;

extracting the root, $x^2 - x - 2 - \frac{1}{2} = \pm \frac{19}{2}$,

and $x^2 - x = 12$, or -7, the former of which only will give a possible value of x; and $\therefore$

completing the square, $x^2 - x + \frac{1}{4} = 12 + \frac{1}{4} = \frac{49}{4}$;

extracting the root, $x - \frac{1}{2} = \pm \frac{7}{2}$,

and $x = 4$, or -3;

consequently the carpets contain 16 and 36 square yards, respectively.

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26. A man playing at hazard won at the first throw, as much money as he had in his pocket; at the second throw, he won 5 shillings more than the square root of what he then had; at the third throw, he won the square of all he then had; and then he had £.112. 16*s*. What had he at first?

Let $x =$ the number of shillings he had at first;
$\therefore 2x =$ the number he had after the first throw,
$\sqrt{2x} + 5 =$ the number won the second throw,
and $2x + \sqrt{2x} + 5 =$ the number he had after the second throw;

also $\overline{2x + \sqrt{2x} + 5}|^2 =$ the number won the third throw;

$\therefore \overline{2x + \sqrt{2x} + 5}|^2 + (2x + \sqrt{2x} + 5) = 2256$;

compl^g. the square, $\overline{2x + \sqrt{2x} + 5}|^2 + (2x + \sqrt{2x} + 5) + \frac{1}{4}$

$$= 2256 + \frac{1}{4} = \frac{9025}{4};$$

extracting the root, $2x + \sqrt{2x} + 5 + \frac{1}{2} = \pm \frac{95}{2}$;

$\therefore 2x + \sqrt{2x} = 42$, or $- 53$, the latter of which gives impossible values of x, and $\therefore 2x + \sqrt{2x} = 42$, to answer the conditions of the problem;

completing the square, $2x + \sqrt{2x} + \frac{1}{4} = 42 + \frac{1}{4} = \frac{169}{4}$;

extracting the root, $\sqrt{2x} + \frac{1}{2} = \pm \frac{13}{2}$;

$$\therefore \sqrt{2x} = 6, \text{ or } -7,$$

$$\text{and } 2x = 36, \text{ or } 49;$$

$$\therefore x = 18, \text{ or } \frac{49}{2};$$

and consequently he began with 18 shillings.

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27. What number is that, which being divided by the product of its two digits, the quotient is 2, and if 27 be added to it, the digits will be inverted?

Let x and y be the digits;

$$\therefore 10x + y = \text{the number},$$

$$\text{and } \frac{10x + y}{xy} = 2;$$

$$\therefore 10x + y = 2xy;$$

$$\text{also } 10x + y + 27 = 10y + x;$$

by transposition, $9x + 27 = 9y$,

$$\text{or } x + 3 = y;$$

which value of y being substituted in the first equation,

$$10x + x + 3 = 2x . (x + 3),$$

$$\text{or } 11x + 3 = 2x^2 + 6x;$$

by transposition, $2x^2 - 5x = 3$,

$$\text{or } x^2 - \frac{5}{2} . x = \frac{3}{2};$$

completing the square, $x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16} = \frac{49}{16}$;

extracting the root, $x - \frac{5}{4} = \pm\frac{7}{4}$,

and $x = 3$, or $-\frac{1}{2}$, the first of which only answers th conditions; $\therefore y = 6$, and the number is 36.

28. There are three numbers, the difference of whose difference is 8; their sum is 41; and the sum of their squares 69[illegible] What are the numbers?

Let $x =$ the second number,
and $y =$ the difference of the second and the least;
$\therefore x - y, x$, and $x + y + 8$ are the numbers,
and their sum $= 3x + 8 = 41$;
by transposition, $3x = 33$,
and $x = 11$;
$\therefore \overline{11 - y}|^2 + 121 + \overline{19 + y}|^2 = 699$,
or $603 + 16y + 2y^2 = 699$;
by transposition, $2y^2 + 16y = 96$,
and $y^2 + 8y = 48$;
completing the square, $y^2 + 8y + 16 = 64$;
extracting the root, $y + 4 = \pm 8$,
and $y = 4$, or -12, both which values answer the conditions; and the numbers are 7, 11, and 23.

29. There are three numbers, the difference of whose differences is 5; their sum is 44; and continual product is 1950. What are the numbers?

Let $x =$ the second number,
and $y =$ the difference of the second and the least;

$\therefore$ the numbers are $x - y$, x, $x + y + 5$,
and $(x - y + x + x + y + 5 =) 3x + 5 = 44$;

by transposition, $3x = 39$,

and $x = 13$;

$$\therefore (13 - y) . 13 . (18 + y) = 1950,$$

$$\text{and } (13 - y) . (18 + y) = 150,$$

$$\text{or } 234 - 5y - y^2 = 150;$$

$$\therefore y^2 + 5y = 84;$$

completing the square, $y^2 + 5y + \frac{25}{4} = 84 + \frac{25}{4} = \frac{361}{4}$;

extracting the root, $y + \frac{5}{2} = \pm \frac{19}{2}$,

and $y = 7$, or -12, both which values answer the conditions; $\therefore$ the numbers are 6, 13, and 25.

30. There were two rows of counters, of which the upper row exceeded the lower by one. A certain number having been taken from the upper row, and as many as then remained from the lower row, it was found that the square of the number remaining in the lower row, added to the square root of that number, is equal to 72 divided by the excess of the number taken from the upper row above unity. Required the number of counters taken from the upper row.

Let $x + 1$ = the number in the upper row;
$\therefore x$ = the number in the lower,
$y + 1$ = the number taken from the upper row;
$\therefore x - y$ = the remainder in the upper row,
and y = the remainder in the lower row;

$$\therefore y^2 + \sqrt{y} = \frac{72}{y},$$

$$\text{and } y^3 + y^{\frac{3}{2}} = 72;$$

completing the square, $y^3 + y^{\frac{3}{2}} + \frac{1}{4} = 72 + \frac{1}{4} = \frac{289}{4}$;

extracting the root, $y^{\frac{3}{2}} + \frac{1}{2} = \pm \frac{17}{2}$;

$$\therefore y^{\frac{3}{2}} = 8, \text{ or } -9,$$
$$\text{and } y^3 = 64, \text{ or } 81;$$

$\therefore y = 4$, or $\sqrt[3]{81}$, the latter of which is excluded by the nature of the question.

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31. A grocer sold 80 pounds of mace, and 100 pounds of cloves for £.65; but he sold 60 pounds more of cloves for £.20, than he did of mace for £.10. What was the price of a pound of each?

Let x = the price of a lb. of mace (in pounds), and y = the price of a lb. of cloves;

then $x : 10 :: 1$: the number of lbs. of mace for £.10 $= \frac{10}{x}$.

In the same way $\frac{20}{y}$ = the number of lbs. of cloves for £.20;

$$\therefore \frac{20}{y} = 60 + \frac{10}{x},$$

$$\text{or } \frac{2}{y} = 6 + \frac{1}{x} = \frac{6x + 1}{x}.$$

$$\text{and } \therefore y = \frac{2x}{6x + 1}.$$

$$\text{Again, } 80x + 100y = 65,$$
$$\text{or } 16x + 20y = 13;$$

$$\therefore 16x + \frac{40x}{6x + 1} = 13,$$

$$\text{and } 96x^2 + 56x = 78x + 13;$$
$$\text{by transposition, } 96x^2 - 22x = 13,$$

$$\text{and } x^2 - \frac{22}{96} \cdot x = \frac{13}{96};$$

completing the square, $x^2 - \frac{22}{96} \cdot x + \overline{\frac{11}{96}}\Big|^2 = \frac{121}{\overline{96}|^2} +$

$\frac{13}{96} = \frac{1369}{\overline{96}|^2}$;

extracting the root, $x - \frac{11}{96} = \pm \frac{37}{96}$;

and $\therefore x = \frac{1}{2}$, or $-\frac{13}{48}$, which last does

not answer the conditions; and $y = \frac{1}{4}$.

$\therefore$ the price of a pound of mace is 10 shillings, and of a pound of cloves, is 5 shillings.

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32. *A* and *B* engage to reap a field for £.4. 10*s.*; and as *A* alone could reap it in 9 days, they promise to complete it in 5 days. They found however that they were obliged to call in *C*, an inferior workman, to assist them for the two last days, in consequence of which *B* received 3*s.* 9*d.* less than he otherwise would have done. In what time could *B* or *C*, alone reap the field?

Let x = the number of days in which *B* could reap the field, and y = the number in which *C* could reap it;

then $\frac{1}{9} + \frac{1}{x} : \frac{1}{x} :: 90$: the number of shillings *B* would

have received $= \frac{810}{9 + x}$,

and $\frac{5}{x} \times 90 = \frac{450}{x}$ = the number he did receive;

$$\therefore \frac{810}{9+x} - \frac{450}{x} = 3\tfrac{3}{4} = \frac{15}{4},$$

$$\text{or } \frac{54}{9+x} - \frac{30}{x} = \frac{1}{4};$$

$\therefore 216x - 1080 - 120x = 9x + x^2$;

by transposition, $x^2 - 87x = -1080$;

completing the square, $x^2 - 87x + \frac{7569}{4} = \frac{7569}{4} - 1080 = \frac{3249}{4}$;

extracting the root, $x - \frac{87}{2} = \pm \frac{57}{2}$,

and $x = 72$, or 15.

Let $x = 15$, then $\frac{5}{9} + \frac{5}{15} + \frac{2}{y} = 1$;

by transposition, $\frac{2}{y} = 1 - \frac{5}{9} - \frac{1}{3} = \frac{1}{9}$;

$\therefore y = 18$ the number of days in which C could reap the field. The other value of x is excluded by the nature of the question.

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33. Throwing out the three court cards from a suit of spades, and placing the remainder in two heaps, I find the sum of the pips in the smaller heap is to the sum in the greater, as the number of cards in the greater heap is to the number of cards in the smaller. But if I add the seventh card to the smaller heap, the difference of the number of pips in the two heaps is equal the square of the number of cards in the smaller. Required the number of pips and cards in each.

The whole number of cards $= 10$,

and the whole number of pips $= 55$.

If $\therefore x =$ the number of cards in the larger heap;

$10 - x =$ the number in the smaller,

and if $y =$ the number of pips in the smaller;

$55 - y =$ the number in the larger;

$\therefore y : 55 - y :: x : 10 - x$,

and (*Alg.* 179.) $y : 55 :: x : 10$;

or $y : 11 :: x : 2$;

$\therefore\ 2y = 11x$.

Again, after the change, $y + 7$ = the number of pips in the smaller heap,

and $55 - y - 7 = 48 - y$ = the number of pips in the larger heap,

and $\therefore$ their difference $= 2y - 41 = \overline{11 - x}|^2$,

and by substituting for $2y$ its value, $11x - 41 = \overline{11 - x}|^2$ $= 121 - 22x + x^2$,

and $\therefore$ by transposition, $x^2 - 33x = -162$;

completing the square, $x^2 - 33x + \overline{\frac{33}{2}}|^2 = \frac{1089}{4} - 162 = \frac{441}{4}$;

extracting the root, $x - \frac{33}{2} = \pm \frac{21}{2}$,

and $x = 6$, or 27; but 27 being inconsistent with the nature of the problem, the number of cards in the larger heap $= 6$, and $\therefore$ the number in the smaller heap $= 4$; and the number of pips in the smaller $= \frac{11x}{2} = 33$,

and $\therefore$ the number in the greater heap $= 22$.

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34. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the periphery of each wheel be increased one yard, it will make only 4 revolutions more than the hind-wheel in the same space. Required the circumference of each.

Let x = the number of yards in the circumference of the larger,

and y = the number in the circumference of the less;

$$\text{then } \frac{120}{x} = \frac{120}{y} - 6,$$

$$\text{or } 20y = 20x - xy;$$

$$\therefore \text{ by transposition, } xy = 20x - 20y.$$

$$\text{Again, } \frac{120}{x+1} = \frac{120}{y+1} - 4,$$

$$\text{or } 30 . (y+1) = (x+1) . (29 - y,)$$

$$\text{or } 30y + 30 = 29x + 29 - xy - y;$$

$$\text{by transposition, } xy = 29x - 1 - 31y,$$

$$\text{and } \therefore 29x - 1 - 31y = 20x - 20y;$$

$$\text{by transposition, } 9x = 11y + 1,$$

$$\text{or } x = \frac{11y+1}{9};$$

$$\therefore \text{ by substitution, } \frac{11y^2 + y}{9} = \frac{220y + 20}{9} - 20y,$$

$$\text{or } 11y^2 + y = 220y + 20 - 180y = 40y + 20;$$

$$\therefore y^2 - \frac{39}{11} . y = \frac{20}{11};$$

$$\text{completing the square, } y^2 - \frac{39}{11} . y + \overline{\frac{39}{22}}\Big|^2 = \frac{1521}{\overline{22}|^2} + \frac{20}{11} = \frac{2401}{\overline{22}|^2};$$

$$\text{extracting the root, } y - \frac{39}{22} = \pm \frac{49}{22};$$

$$\therefore y = 4, \text{ or } -\frac{5}{11};$$

$\therefore$ the number of yards in the circumference of the less $= 4$, and the number in the circumference of the greater $= \frac{11y+1}{9} = 5$.

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35. On the late jubilee, a gentleman treated his tenantry at the following rate. He allowed for each poor child a certain number of sixpences, for each poor woman sixpence more, and for each poor man sixpence still in addition. The number

of women was one-fourth greater than the number of men; the number of children was equal to twice the square of the difference between the numbers of men and women; and the whole expence was £.8. 2*s*. But had each child been allowed as much as each woman, the expence on their account added to nine times the difference of what the men and women cost, would have been £.4. 18*s*. Required the number of men, women, and children, and the allotment to each.

Let $4x =$ the number of men;
$\therefore\ 5x =$ the number of women,
and $2x^2 =$ the number of children.
Let $y =$ the number of sixpences each child had;
$\therefore\ y+1 =$ the number each woman had,
and $y+2 =$ the number each man had;

$$\therefore\ 2x^2y + 9xy + 13x = 324;$$
$$\text{also } 2x^2y + 2x^2 + 9xy - 27x = 196;$$

$$\therefore \text{ by subtraction, } 2x^2 - 40x = -128,$$
$$\text{and } x^2 - 20x = -64;$$

completing the square, $x^2 - 20x + 100 = 100 - 64 = 36$;
extracting the root, $x - 10 = \pm 6$,
and $x = 16$, or 4, the former of which will not answer the conditions of the problem; $\therefore$ the number of men was 16, of women 20, and of children 32.

$$\text{Also } 32y + 36y + 52 = 324,$$
$$\text{or } 68y = 272;$$
$$\therefore\ y = 4;$$

$\therefore$ each child had 2 shillings, each woman 2*s*. 6*d*, and each man 3*s*.

36. *A* and *B* were going to market, the first with cucumbers, and the second with three times as many eggs; and they find that if *B* gave all his eggs for the cucumbers, *A* would lose 10 pence, according to the rate at which they were then

selling. *A* therefore reserves two-fifths of his cucumbers; by which *B* would lose sixpence, according to the same rate. But *B*, selling the cucumbers at sixpence apiece, gains upon the whole the price of 6 eggs. Required the number of eggs and cucumbers, and their price.

Let $x =$ the number of cucumbers,
and $y =$ the price of one;
$\therefore 3x =$ the number of eggs,

and $\frac{xy - 10}{3x} =$ the price of one egg;

$$\text{also } \frac{3}{5} xy = xy - 16,$$

$$\text{or } 3xy = 5xy - 80;$$

$$\therefore 2xy = 80,$$

$$\text{and } xy = 40;$$

$$\text{also } \frac{3}{5} . 6x - (xy - 10) = \frac{2 . (xy - 10)}{x},$$

$$\text{or } \frac{18}{5} x^2 - 30x = 60,$$

$$\text{and } 9x^2 - 75x = 150;$$

completing the square, $9x^2 - 75x + \overline{\frac{75}{6}}\Big|^2 = \frac{5625}{36} + 150$

$= \frac{11025}{36}$;

$$\text{extracting the root, } 3x - \frac{75}{6} = \pm \frac{105}{6},$$

and $3x = 30$, or -5, the latter of which is excluded by the nature of the problem; $\therefore x = 10$,

$$\text{and } y = \frac{40}{x} = 4.$$

Hence the number of eggs was 30, and of cucumbers 10; $\therefore$ the price of a cucumber was 4 pence, and of an egg $= \frac{xy - 10}{3x} = 1$ penny.

37. A person bought a certain number of larks and sparrows for 6 shillings. He gave as many pence *per* dozen for larks as there were sparrows, and as many pence *per* score for sparrows as there were larks. If he had bought 10 more of each, (the price of larks remaining the same) and had given as much *per* dozen for sparrows as he gave *per* score for larks, they would have cost £.1. 5*s*. 5*d*. Required the number of each.

Let x = the number of larks, and ∴ = the number of pence *per* score for sparrows,

y = the number of sparrows, and ∴ = the number of pence *per* dozen for larks;

$$\therefore \left(\frac{xy}{12} + \frac{xy}{20} =\right) \frac{2\,xy}{15} = 72,$$

$$\text{and } xy = 15 \times 36 = 540.$$

Again, if $x + 10$ = the number of larks,

and $y + 10$ = the number of sparrows;

then the price of the larks $= y \times \dfrac{x + 10}{12} = \dfrac{xy + 10y}{12}$,

$= 10 . \dfrac{54 + y}{12}$,

and the price *per* dozen of sparrows $= \dfrac{y}{12} \times 20 = \dfrac{5\,y}{3}$;

$$\therefore \text{ the price of the sparrows} = \frac{5\,y^2 + 50\,y}{3 \times 12},$$

$$\text{and } \frac{54 + y}{12} \times 10 + \frac{5 . (y^2 + 10y)}{3 \times 12} = 305,$$

$$\text{and } (54 + y)30 + 5 . (y^2 + 10\,y) = 36 \times 305,$$

$$\text{or } y^2 + 16\,y + 324 = 36 \times 61 = 2196;$$

by transposition, $y^2 + 16\,y = 1872$;

completing the square, $y^2 + 16y + 64 = 1872 + 64 = 1936$;

extracting the root, $y + 8 = \pm 44$;

∴ $y = 36$, or -52, the latter of which will not answer the conditions of the problem,

$$\text{and } x = \frac{15 \times 36}{y} = 15;$$

$\therefore$ he bought 15 larks, and 36 sparrows.

38. A poulterer bought a certain number of ducks and 18 turkeys for £.5. 10*s.*; each turkey costing within one shilling as much as three ducks. He afterwards bought as many ducks and 5 over, and 20 turkeys, giving one shilling a piece more for each duck and turkey than before; and finds that the value of his former purchase is to the value of the present one :: 2 : 3. Required the number of ducks, and the prices of the ducks and turkeys at the first purchase.

Let x = the number of ducks required,
and y = the price of a duck;
$\therefore 3y - 1$ = the price of a turkey,

$$\text{and } xy + 54y - 18 = 110,$$
$$\text{or } xy + 54y = 128.$$

Now at the second purchase, $x+5$ = the number of ducks, $y+1$, and $3y$ = the price of a duck and turkey, respectively;

$$\therefore 110 : xy + x + 65y + 5 :: 2 : 3,$$
$$\text{or } 55 : xy + x + 65y + 5 :: 1 : 3;$$
$$\therefore xy + x + 65y + 5 = 165,$$
$$\text{and } xy + x + 65y = 160;$$
$$\text{but } xy + 54y = 128;$$
$$\therefore \text{by subtraction, } x + 11y = 32,$$

or $x = 32 - 11y$, which being substituted in the first equation,

$$32y - 11y^2 + 54y = 128,$$
$$\text{or } 11y^2 - 86y = -128,$$
$$\text{and } y^2 - \frac{86}{11} \cdot y = -\frac{128}{11};$$

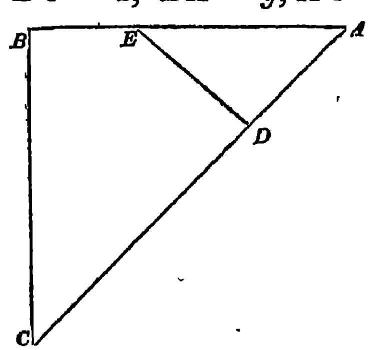
completing the square, $y^2 - \frac{86}{11} \cdot y + \overline{\frac{43}{11}}\bigg|^2 = \frac{1849}{121} - \frac{128}{11} = \frac{441}{121}$;

extracting the root, $y - \frac{43}{11} = \pm \frac{21}{11}$,

and $y = \frac{64}{11}$, or 2, the former which makes x negative, and $\therefore$ the price of a duck is 2 shillings, and the price of a turkey = 5 shillings. Also the number of ducks $= 32 - 11y = 10$.

39. There are three towns A, B, and C; the road from B to A forming a right angle with that from B to C. Now a person has to go from B to A, but after travelling a certain distance towards A, he crosses over by the nearest way to the road which leads from C to A, and when on this road he is 3 miles from A and 7 from C. He then proceeds to A, and when arrived there he finds that he has gone a distance, equal to one-fourth of the distance from B to C, more than he would have done, had he gone the direct road from B to A. Required the distance of B from A and C.

Let $BC = x$, $BA = y$, $AC = 10$,



and since the shortest path from a given point to a given straight line is a perpendicular drawn from that point,

draw ED perpendicular to AC; E being the point where he leaves the road BA; $\therefore$ D is the point where he enters the road CA; $\therefore$ $CD = 7$, and $DA = 3$.

By similar $\triangle$'s $BA : CA :: DA : AE$,

$$\text{or } y : 10 :: 3 : AE = \frac{30}{y},$$

$$\text{and } BA : BC :: DA : DE,$$

$$\text{or } y : x :: 3 : DE = \frac{3x}{y};$$

$$\therefore \frac{3x}{y} + 3 = \frac{30}{y} + \frac{x}{4},$$

$$\text{or } 3 . (x + y) = 30 + \frac{1}{4} xy,$$

$$\text{and } 2xy - 24 . (x + y) = -240;$$

$$\text{but } x^2 + y^2 = 100;$$

$\therefore$ by addition, $x^2 + 2xy + y^2 - 24 . (x + y) = -140$;

completing the square, $\overline{x + y}|^2 - 24 . (x + y) + 144 = 144 - 140 = 4$;

extracting the root, $x + y - 12 = \pm 2$;

$\therefore x + y = 14$, or 10, the former of which only answers the conditions, $\therefore x^2 + 2xy + y^2 = 196$,

$$\text{but } x^2 + y^2 = 100;$$

$$\therefore \text{ by subtraction, } 2xy = 96,$$

$$\text{and } x^2 - 2xy + y^2 = 4,$$

$$\text{whence } x - y = \pm 2,$$

$$\text{but } x + y = 14;$$

$$\therefore \text{ by addition, } 2x = 16, \text{ or } 12,$$

$$\text{and by subtraction, } 2y = 12, \text{ or } 16;$$

$$\therefore x = 8, \text{ or } 6,$$

$$\text{and } y = 6, \text{ or } 8;$$

$\therefore$ the distance of B from C is 8, or 6 miles, and from A is 6, or 8 miles.

40. In a garden is a square bowling-green, a side of which is 30 yards, and near to it is a rectangular grass-plot. The number of square yards in the area of the grass-plot is a mean proportional between $\frac{192}{79}$, and the number of square yards contained in the grass-plot and bowling-green together. Also the number of square yards contained in the square described on the diameter of the grass-plot is a mean proportional between 10, and the number of square yards contained in the aforesaid square increased by the number contained in the bowling-green. Required the area and sides of the grass-plot.

Let x and y = the number of yards in the sides;

$$\therefore xy = \text{the area,}$$

$$\text{and } \frac{192}{79} : xy :: xy : xy + 900;$$

$$\therefore x^2y^2 = \frac{192}{79} \cdot xy + \frac{172800}{79};$$

$$\text{by transposition, } x^2y^2 - \frac{192}{79}xy = \frac{172800}{79};$$

$$\text{completing the square, } x^2y^2 - \frac{192}{79}xy + \overline{\frac{96}{79}}\Big|^2 = \frac{172800}{79} + \overline{\frac{96}{79}}\Big|^2$$

$$= \frac{13660416}{\overline{79}|^2};$$

$$\text{extracting the root, } xy - \frac{96}{79} = \pm \frac{3696}{79};$$

$$\therefore xy = 48, \text{ or } -\frac{3600}{79}, \text{ the former of which only}$$

answers the conditions of the problem.

Again, $10 : x^2 + y^2 :: x^2 + y^2 : x^2 + y^2 + 900$;

$$\therefore \overline{x^2 + y^2}|^2 = 10 \cdot (x^2 + y^2) + 9000;$$

by transposition, $\overline{x^2 + y^2}|^2 - 10 \cdot (x^2 + y^2) = 9000$;

completing the square, $\overline{x^2 + y^2}|^2 - 10 \cdot (x^2 + y^2) + 25 = 9025$;

$$\text{extracting the root, } x^2 + y^2 - 5 = 95;$$

$\therefore\ x^2+y^2=100$, or -90, the latter of which will not answer the conditions;

but, $2xy = 96$;

$\therefore$ by addition, $x^2+2xy+y^2=196$, and $x+y=\pm 14$;
by subtraction, $x^2-2xy+y^2=4$, and $x-y=\pm 2$;

$\therefore$ by addition, $2x=\pm 16$, or ± 12;

$\therefore\ x=\pm 8$, or ± 6;

by subtraction, $2y=\pm 12$, or ± 16;

$\therefore\ y=\pm 6$, or ± 8;

but the positive values will only answer the conditions, and the area is 48 square yards.

---

41. A rectangular vat, 3 feet deep, when filled to the depth of 2 feet, holds less than when completely filled, by a number of cubic feet equal to 24, together with half the number of feet in the perimeter of the base. It is also observed, that the length of a pole, which reaches from one of the corners of the top to the opposite corner of the bottom of the vat, is equal to one-eighth of the number of feet in the square inscribed on the diagonal of the bottom. Required the dimensions of the vat.

Let x and y be the number of feet in the sides of the base;

then $(3\,xy - 2\,xy =)\ xy = 24 + x + y$,

and $\sqrt{9+x^2+y^2} = \frac{1}{8}\cdot(x^2+y^2)$;

$\therefore\ x^2+y^2-8\sqrt{9+x^2+y^2}=0$,

and $(9+x^2+y^2)-8\sqrt{9+x^2+y^2}=9$;

completing the square, $(9+x^2+y^2)-8\sqrt{9+x^2+y^2}+16=25$;

extracting the root, $\sqrt{9+x^2+y^2}-4=\pm 5$,

and $\sqrt{9+x^2+y^2}=9$, or -1;

$\therefore\ 9+x^2+y^2=81$, or 1;

by transposition, $x^2 + y^2 = 72$, or -8, the latter of which is impossible;

$$\text{but } 2xy = 48 + 2.(x+y);$$

$\therefore$ by addition, $x^2 + 2xy + y^2 = 120 + 2.(x+y)$;

by transposition, $\overline{x+y}|^2 - 2.(x+y) = 120$;

completing the square, $\overline{x+y}|^2 - 2.(x+y) + 1 = 121$;

extracting the root, $x + y - 1 = \pm 11$,

and $x + y = 12$, or -10, the latter of which is impossible;

$$\therefore x^2 + 2xy + y^2 = 144;$$
$$\text{but } 4xy = 144;$$

$\therefore$ by subtraction, $x^2 - 2xy + y^2 = 0$,

$$\text{and } x - y = 0;$$
$$\text{but } x + y = 12;$$

$\therefore$ by addition, $2x = 12$,

$$\text{and } x = 6;$$

$\therefore y = x = 6$, and the base is a square whose side is 6 feet.

---

42. A person bought two cubical stacks of hay for £.41, each of which cost as many shillings *per* solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each?

Let x = the number of yards in a side of the larger, and y = the number in a side of the less;

then x^3 and y^3 = the number of solid yards in the stacks, and x^2, and y^2 = the number of square yards in their bases;

$$\therefore x^2 - y^2 = 9,$$
$$\text{and } x^3y + y^3x = 820;$$
$$\therefore x^2 + y^2 = \frac{820}{xy},$$

$$\text{and } x^4 + 2x^2y^2 + y^4 = \frac{\overline{820}|^2}{x^2y^2};$$

$$\text{but } x^4 - 2x^2y^2 + y^4 = 81;$$

$$\therefore \text{ by subtraction, } 4x^2y^2 = \frac{\overline{820}|^2}{x^2y^2} - 81,$$

$$\text{or } x^4y^4 = \frac{\overline{820}|^2}{4} - \frac{81x^2y^2}{4};$$

$$\text{by transposition, } x^4y^4 + \frac{81}{4}.x^2y^2 = \overline{\frac{820}{2}}\Big|^2 = \overline{410}|^2 = 168100;$$

$$\text{completing the square, } x^4y^4 + \frac{81}{4}.x^2y^2 + \overline{\frac{81}{8}}\Big|^2 = 168100$$

$$+ \frac{6561}{64} = \frac{10764961}{64};$$

$$\text{extracting the root, } x^2y^2 + \frac{81}{8} = \pm\frac{3281}{8};$$

$\therefore x^2y^2 = 400$, or $-\frac{1681}{4}$, the latter of which is impossible;

$\therefore xy = \pm 20$, the positive value only answering the conditions of the problem;

$$\text{and } x^2 + y^2 = \frac{820}{xy} = \frac{820}{20} = 41;$$

$$\text{but } 2xy = 40;$$

$\therefore$ by addition, $x^2+2xy+y^2 = 81$, and $x+y = \pm 9$,
and by subtraction, $x^2-2xy+y^2=1$; $\therefore x-y = \pm 1$;

$\therefore$ by addition, $2x = \pm 10$, or ± 8,
and $x = \pm 5$, or ± 4;
by subtraction, $2y = \pm 8$, or ± 10,
and $y = \pm 4$, or ± 5;

$\therefore$ the prices were £.25, and £.16.

---

43. *A* and *B* put out different sums to interest, amounting together to £.200. *B*'s rate of interest was £.1. *per* cent.

more than *A*'s. At the end of 5 years, *B*'s accumulated simple interest wanted but £.4 to be double of *A*'s. At the end of 10 years, *A*'s principal and interest was to *B*'s as 5 : 8. Required the separate sums put out by each, and the rate *per* cent.

$$\text{Let } 4x = A\text{'s money (in pounds)};$$
$$\therefore 4.(50 - x) = B\text{'s money};$$
$$y = A\text{'s rate of interest};$$
$$\therefore y + 1 = B\text{'s rate};$$
$$\therefore \frac{xy}{5} = A\text{'s interest after 5 years},$$
$$\text{and } \frac{(50 - x).(y + 1)}{5} = B\text{'s interest after 5 years};$$
$$\therefore \frac{(50 - x).(y + 1)}{5} + 4 = 2.\frac{xy}{5},$$
$$\text{or } 50y - xy + 50 - x + 20 = 2xy;$$
$$\therefore 3xy = 50y + 70 - x \quad . \quad . \quad . \quad . \quad [1].$$

Again, after 10 years, *A*'s capital and interest $= 4x + \frac{2xy}{5} = 2.\frac{10x + xy}{5}$,

$$\text{and } B\text{'s} = 2.\frac{(50 - x).(y + 11)}{5};$$
$$\therefore 2.\frac{10x + xy}{5} : 2.\frac{(50 - x).(y + 11)}{5} :: 5 : 8;$$
$$\therefore 80x + 8xy = 250y - 5xy + 2750 - 55x;$$

by transposition, $13xy = 250y + 2750 - 135x$. [2]

but from [1] $15xy = 250y + 350 - 5x$;

$\therefore$ by subtraction, $2xy = 130x - 2400$ . . . [3]

multiplying [1] by 13, and [2] by 3;

$$\therefore 650y + 910 - 13x = (39xy =)\, 750y + 8250 - 405x;$$

by transposition, $100y = 392x - 7340$,

or $50y = 196x - 3670$, which being multiplied by x, $50xy = 196x^2 - 3670x$; but [3] being multiplied by 25,

$$50xy = 3250x - 60000;$$

$$\therefore 196x^2 - 3670x = 3250x - 60000;$$

by transposition, $196x^2 - 6920x = -60000;$

completing the square, $196x^2 - 6920x + \overline{\frac{1730}{7}}\Big|^2$

$$= \frac{2992900}{49} - 60000 = \frac{52900}{49};$$

extracting the root, $14x - \frac{1730}{7} = \pm\frac{230}{7};$

$$\therefore 14x = 280, \text{ or } \frac{1500}{7};$$

$$\therefore x = 20;$$

$\therefore$ *A*'s money $= 4x = 80$ pounds, and *B*'s$=120$ pounds,

and $y = \frac{196 \times 20 - 3670}{50} = 5;$

$\therefore$ *A*'s rate of interest was 5 *per* cent., and *B*'s 6 *per* cent.

44. When the price of brandy was three times the price of British spirit, a merchant made two mixtures of brandy and British spirit, and the prices *per* gallon were in the ratio of 9 to 10. He afterwards mixed twice as much brandy with the same quantity of British spirit in each case, and the relative price was the same as before. Required the ratio of the quantities mixed.

Suppose at first x gallons of British spirit were mixed with one gallon of brandy, in one case; and y gallons of British spirit with one gallon of brandy in the other case; then

$$\left.\begin{aligned} x+1 : 1 &:: x+3 : \frac{x+3}{x+1} \\ \text{and } y+1 : 1 &:: y+3 : \frac{y+3}{y+1} \end{aligned}\right\} \text{the relative prices of the first mixtures per gallon;}$$

$$\text{hence, } \frac{x+3}{x+1} : \frac{y+3}{y+1} :: 9 : 10.$$

In the same manner $\frac{x+6}{x+2} : \frac{y+6}{y+2} :: 9 : 10$,

and $\therefore xy + x + 3y + 3 : xy + 3x + y + 3 :: 9 : 10$,

div^{do}. $xy + x + 3y + 3 : 2x - 2y :: 9 : 1$;

$\therefore xy + x + 3y + 3 = 18x - 18y$;

by transposition, $xy - 17x + 21y + 3 = 0$;

also $xy + 2x + 6y + 12 : xy + 6x + 2y + 12 :: 9 : 10$;

div^{do}. $xy + 2x + 6y + 12 : 4x - 4y :: 9 : 1$;

$\therefore xy + 2x + 6y + 12 = 36x - 36y$;

by transposition, $xy - 34x + 42y + 12 = 0$;

but $xy - 17x + 21y + 3 = 0$;

$\therefore$ by subtraction, $17x - 21y - 9 = 0$;

also $2xy - 34x + 42y + 6 = 0$,

and $xy - 34x + 42y + 12 = 0$;

$\therefore$ by subtraction, $xy - 6 = 0$,

or $xy = 6$.

Now $x = \frac{21y + 9}{17}$;

$\therefore \frac{21y^2 + 9y}{17} = (xy =) 6$,

and $21y^2 + 9y = 102$,

or $y^2 + \frac{3}{7}y = \frac{34}{7}$;

completing the square, $y^2 + \frac{3}{7}y + \frac{9}{196} = \frac{34}{7} + \frac{9}{196} = \frac{961}{196}$;

extracting the root, $y + \frac{3}{14} = \pm \frac{31}{14}$;

$\therefore y = 2$, or $- \frac{17}{7}$;

$\therefore x = 3$;

$\therefore$ the first mixtures were in the ratio of 3 to 1, and 2 to 1; and the second in the ratio of 3 to 2, and of equality.

Sect. X.

Examples of the Solution of Problems in Arithmetical and Geometrical Progressions.

1. A person bought 7 books, the particular prices of which (in shillings) were in arithmetical progression. The price of the next above the cheapest was 8 shillings, and the price of the dearest 23 shillings. What was the price of each book?

$$\begin{aligned} \text{Let } x &= \text{the price of the cheapest,} \\ \text{and } y &= \text{the common difference;} \\ \text{then } x + y &= \text{the price of the second} = 8, \\ \text{and } x + 6y &= \text{the price of the dearest} = 23; \end{aligned}$$

$$\therefore \text{ by subtraction, } 5y = 15,$$

$$\text{and } y = 3;$$

$$\therefore x = 8 - y = 8 - 3 = 5,$$

and $\therefore$ the prices are 5, 8, 11, 14, 17, 20, 23 shillings, respectively.

2. A number consists of 3 digits, which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits will be inverted. Required the number.

$$\text{Let } \left.\begin{array}{l} x - y \\ x \\ x + y \end{array}\right\} \text{ represent the digits;}$$

then the number will be $100.(x-y)+10x+x+y = 111x - 99y$;

$$\therefore \frac{111x - 99y}{3x} = 26,$$

$$\text{or } 37x - 33y = 26x,$$

$$\text{and } \therefore \text{ by transposition, } 11x = 33y,$$

$$\text{and } x = 3y.$$

Again, $111x - 99y + 198 = 100.(x+y) + 10x + (x-y) = 111x + 99y$;

$$\therefore \text{ by transposition, } 198y = 198,$$

$$\text{and } y = 1;$$

$$\therefore x = 3y = 3;$$

$\therefore$ the digits are 2, 3, and 4, and the number = 234.

---

3. The sum of £.1. 7*s.* was to be raised by subscription by three persons *A, B,* and *C*; the sums to be subscribed by them respectively forming an arithmetical progression. But *C* dying before the money was paid, the whole fell to *A* and *B*; and *C*'s share was raised between them in the proportion of 3 : 2, when it appeared that the whole sum subscribed by *A* was to the whole sum subscribed by *B* :: 4 : 5. Required the original subscription of *A, B,* and *C.*

Let $x - y$, x, $x+y$, be the respective subscriptions of *A, B,* and *C*;

$$\text{then } 3x = 27; \text{ and } \therefore x = 9.$$

Now $5 : 2 ::$ (*C*'s share =) $9 + y$: the part paid by *B* $= \frac{2}{5}.(9+y)$,

and $5 : 3 :: 9 + y$: the part paid by $A = \frac{3}{5}.(9+y)$,

and consequently, A paid upon the whole $9 - y + \frac{3}{5}$.

$$(9 + y) = \frac{72 - 2y}{5};$$

also B paid upon the whole $9 + \frac{2}{5}(9+y) = \frac{63+2y}{5}$;

$$\text{hence, } 72 - 2y : 63 + 2y :: 4 : 5,$$
$$\text{and } (\textit{Alg. } 179.)\ 135 : 63 + 2y :: 9 : 5,$$
$$\text{and } (\textit{Alg. } 186.)\ 15 : 63 + 2y :: 1 : 5;$$
$$\therefore (21)\ 63 + 2y = 75;$$
$$\text{by transposition, } 2y = 12,$$
$$\text{and } y = 6;$$

$\therefore$ the sums to be subscribed originally were 3, 9, and 15 shillings.

---

4. Four numbers are in arithmetical progression. The sum of their squares is equal to 276, and the sum of the numbers themselves is equal to 32. What are the numbers?

$$\text{Let } 2y = \text{the common difference,}$$

$$\text{and } \left.\begin{array}{l} x + 3y \\ x + y \\ x - y \\ x - 3y \end{array}\right\} \text{be the numbers;}$$

$$\text{then their sum} = 4x = 32,$$
$$\text{and } \therefore x = 8;$$

also the sum of their squares $= 4x^2 + 20y^2 = 276$, in which substituting the value of x found above,

$$256 + 20y^2 = 276;$$
$$\text{by transposition, } 20y^2 = 20;$$
$$\therefore y^2 = 1;$$
$$\text{and } y = \pm 1;$$

hence the numbers are 11, 9, 7, 5.

5. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

Supposing as before, $x + 3y$, $x + y$, $x - y$, and $x - 3y$, to be the numbers;

$$\text{then } 2x^2 + 18y^2 = 200,$$
$$\text{and } 2x^2 + 2y^2 = 136;$$
$$\therefore \text{ by subtraction, } 16y^2 = 64,$$
$$\text{and } 4y = \pm 8;$$
$$\therefore y = \pm 2;$$
$$\text{whence } x^2 = 68 - y^2 = 68 - 4 = 64,$$
$$\text{and } x = \pm 8,$$

and $\therefore$ the numbers are ± 14, ± 10, ± 6, ± 2.

6. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth is 60. Required the numbers.

Let x, xy, xy^2, xy^3, be the numbers;

$$\therefore x + xy = 15,$$
$$\text{and } xy^2 + xy^3 = 60,$$
$$\text{or } y^2 . (x + xy) = 60,$$
$$\text{or } 15y^2 = 60;$$
$$\therefore y^2 = 4,$$
$$\text{and } y = \pm 2,$$
$$\text{and } (x + 2x =) \, 3x = 15;$$
$$\therefore x = 5,$$

and the numbers are 5, 10, 20, 40.

7. The sum of four numbers in geometrical progression is equal to the common ratio + 1; and the first term $= \frac{1}{17}$. Required the numbers.

Let x = the common ratio;

$$\therefore \text{ the numbers are } \frac{1}{17}, \frac{x}{17}, \frac{x^2}{17}, \frac{x^3}{17},$$

$$\text{and } 1 + x = \frac{1 + x + x^2 + x^3}{17} = \frac{(1 + x^2) \cdot (1 + x)}{17},$$

$$\text{and } 1 = \frac{1 + x^2}{17};$$

$$\therefore 17 = 1 + x^2,$$

$$\text{and } 16 = x^2$$

$$\therefore \pm 4 = x,$$

$$\text{and the numbers are } \frac{1}{17}, \frac{4}{17}, \frac{16}{17}, \frac{64}{17}.$$

8. A regiment of militia was just sufficient to form an equilateral wedge. It was afterwards doubled by the supplementary, but was still found to want 385 men to complete a square containing 5 more men in a side, than in a side of the wedge. How many did the regiment at first contain?

Let x = the number of men in a side of the wedge;

$\therefore$ (*Alg*. 212.) $(x + 1) \cdot \frac{x}{2}$ = the number of men in the wedge;

$$\therefore (x + 1) \cdot x + 385 = \overline{x + 5}|^2,$$

$$\text{or } x^2 + x + 385 = x^2 + 10x + 25;$$

$$\therefore \text{ by transposition } 360 = 9x,$$

$$\text{and } 40 = x;$$

$$\therefore \text{ the number of men } = 820.$$

9. After *A* who travelled at the rate of 4 miles an hour, had been set out two hours and three-quarters, *B* set out to overtake him, and in order thereto went four miles and a half the first hour, four and three-quarters the second, five the third; and so on, gaining a quarter of a mile every hour. In how many hours would he overtake *A*?

Let x = the number of hours;

$\therefore$ (*Alg.* 212.) $\left(9 + (x - 1)\frac{1}{4}\right) \times \frac{x}{2}$ = the whole number of miles he travelled;

but $11 + 4x$ = the whole number A travelled;

$$\therefore \left(9 + \frac{1}{4}(x - 1)\right) \cdot \frac{x}{2} = 11 + 4x,$$

$$\text{or } 9x + \frac{x^2}{4} - \frac{x}{4} = 22 + 8x,$$

and by transposition, $\frac{x^2}{4} + \frac{3x}{4} = 22$;

completing the square, $\frac{x^2}{4} + \frac{3x}{4} + \frac{9}{16} = 22 + \frac{9}{16} = \frac{361}{16}$;

extracting the root, $\frac{x}{2} + \frac{3}{4} = \pm \frac{19}{4}$;

$$\therefore \frac{x}{2} = 4, \text{ or } -\frac{11}{2};$$

$$\therefore x = 8, \text{ or } -11;$$

hence in 8 hours he would overtake him. -11 not answering the conditions of the problem.

10. The base of a right-angled triangle is 6, and the sides are in arithmetical progression; it is required to find the other two sides.

Let $6 - x$, 6, and $6 + x$ be the sides;

then $36 - 12x + x^2 + 36 = 36 + 12x + x^2$ (*Eucl.* B. I. p. 48.)

by transposition, $24x = 36$,

$$\text{and } x = \frac{3}{2};$$

$$\therefore \text{ the sides are } \frac{9}{2}, \frac{12}{2}, \text{ and } \frac{15}{2}.$$

But if 6 be the first term of the progression; let,

$6, \; 6 + x, \; 6 + 2x$ be the sides;

then $36 + 24x + 4x^2 = 36 + 36 + 12x + x^2$;

by transposition, $3x^2 + 12x = 36$,

or $x^2 + 4x = 12$;

completing the square, $x^2 + 4x + 4 = 16$;

extracting the root, $x + 2 = \pm 4$,

and $x = 2$, or -6,

and the sides are 6, 8, and 10; or 0, 6, and -6. The problem is not properly restricted; the algebraical expression, in this instance, is more precise than the language in which the problem is stated.

11. *A* and *B* set out from London at the same time, to go round the world (23661 miles) one going East, the other West. *A* goes one mile the first day, two the second, and so on. *B* goes 20 miles a day. In how many days will they meet; and how many miles will be travelled by each?

Let x = the number of days;

then (*Alg.* 212.) $(x + 1) \cdot \frac{x}{2}$ = the number of miles *A* goes,

and $20x$ = the number *B* goes;

$$\therefore \frac{x^2 + x}{2} + 20x = 23661,$$

and $x^2 + 41x = 47322$;

completing the square, $x^2 + 41x + \overline{\frac{41}{2}}\Big|^2 = 47322 + \frac{1681}{4} = \frac{190969}{4}$;

$$\therefore \text{ extracting the root, } x + \frac{41}{2} = \pm \frac{437}{2};$$

$$\therefore x = 198, \text{ or } -239;$$

$\therefore$ they travel 198 days; *A* goes 19701, and *B* 3960 miles.

12. A traveller sets out for a certain place, and travels one mile the first day, two the second, and so on. In 5 days afterwards another sets out, and travels 12 miles a day. How long and how far must he travel to overtake the first?

Let x = the number of days;
then $x + 5$ = the number the first travels,

and ∴ (*Alg.* 212.) $(x + 6) . \frac{x + 5}{2}$ = the distance he travels,

and $12x$ = the distance the second travels;

$$\therefore (x + 6) . \frac{x + 5}{2} = 12x,$$

$$\text{and } x^2 + 11x + 30 = 24x;$$

$$\therefore \text{ by transposition, } x^2 - 13x = -30;$$

completing the square, $x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4}$;

extracting the root, $x - \frac{13}{2} = \pm \frac{7}{2}$,

and $x = 3$, or 10.

∴ they are together at the end of 3, and 10 days after the second sets out; and 36 and 120 miles is the distance travelled.

---

13. *A* and *B*, 165 miles distant from each other, set out with a design to meet; *A* travels one mile the first day, two the second, three the third, and so on; *B* travels 20 miles the first day, 18 the second, 16 the third, and so on; How soon will they meet?

Let x = the number of days required;

then $1 + 2 + 3 + \ldots\ldots + x = (1 + x) . \frac{x}{2}$ = the number of miles *A* travelled,

and $20+18+\ldots\ldots+20-2x+2=(42-2x).\frac{x}{2}=$ the number B travelled;

$$\therefore (43-x)\,.\,\frac{x}{2}=165,$$

$$\text{or } x^2-43x=-330;$$

completing the square, $x^2-43x+\frac{1849}{4}=\frac{1849}{4}-330$ $=\frac{529}{4}$;

$$\text{extracting the root, } x-\frac{43}{2}=\pm\frac{23}{2};$$

$$\therefore x=10, \text{ or } 33.$$

Hence it appears that they meet in 10 days. On the 10th day B travels 2 miles, and the next day he rests; the following day he returns 2 miles; the succeeding day 4, and so on, increasing 2 miles every day; and on the 33d day he again comes up with A, who has been travelling forward, every day's journey being one mile longer than that of the preceding day.

---

14. There are four numbers in arithmetical progression whose continual product is 1680, and common difference is 4. Required the numbers.

Let $x+6$, $x+2$, $x-2$, and $x-6$, be the numbers;

$$\text{then } (x^2-36)\,.\,(x^2-4)=1680,$$

$$\text{or } x^4-40x^2+144=1680;$$

$$\therefore \text{by transposition, } x^4-40x^2=1536;$$

$$\text{completing the square, } x^4-40x^2+400=1936,$$

$$\text{extracting the root, } x^2-20=\pm 44;$$

$$\therefore x^2=64, \text{ or } -24,$$

$$\text{and } x=\pm 8, \text{ or } \pm 2\sqrt{-6},$$

and $\therefore$ the numbers are ± 14, ± 10, ± 6, ± 2; the two other values of x being impossible.

15. The product of five numbers in arithmetical progression is 945, and their sum is 25. Required the numbers.

Let $x + 2y$, $x + y$, x, $x - y$, $x - 2y$, be the numbers;

$$\text{then } 5x = 25, \text{ and } \therefore x = 5;$$

also, $x \cdot (x^2 - y^2) \cdot (x^2 - 4y^2) = 945$, or dividing by $x = 5$,

$$(x^2 - y^2) \cdot (x^2 - 4y^2) = 189,$$

$$\text{or } x^4 - 5x^2y^2 + 4y^4 = 189,$$

$$\text{and } 4y^4 - 125y^2 + 625 = 189;$$

$$\text{by transposition, } 4y^4 - 125y^2 = -436;$$

completing the square, $4y^4 - 125y^2 + \overline{\frac{125}{4}}\Big|^2 = \frac{15625}{16} - 436 = \frac{8649}{16}$;

$$\text{extracting the root, } 2y^2 - \frac{125}{4} = \pm \frac{93}{4};$$

$$\therefore 2y^2 = \frac{109}{2}, \text{ or } 8;$$

$$\therefore y^2 = \frac{109}{4}, \text{ or } 4;$$

$$\therefore y = \pm \frac{\sqrt{109}}{2}, \text{ or } \pm 2,$$

and the numbers are 9, 7, 5, 3, 1.

16. A gentleman divided £.210 among three servants, in geometrical progression; the first had £.90 more than the last. How much had each?

Let xy^2, xy, x = the number of pounds each had;

$$\text{then } xy^2 = x + 90,$$

$$\text{and } x + xy + xy^2 = 210;$$

$$\text{or } 2x + xy + 90 = 210;$$

by transposition, $2x + xy = 120$.

Now from the first equation, $x = \dfrac{90}{y^2 - 1}$,

and from the last, $x = \dfrac{120}{y + 2}$;

$$\therefore \frac{120}{y+2} = \frac{90}{y^2-1},$$

$$\text{or } \frac{4}{y+2} = \frac{3}{y^2-1};$$

$$\therefore 4y^2 - 4 = 3y + 6;$$

by transposition, $4y^2 - 3y = 10$;

completing the square, $4y^2 - 3y + \dfrac{9}{16} = 10 + \dfrac{9}{16} = \dfrac{169}{16}$;

extracting the root, $2y - \dfrac{3}{4} = \pm\dfrac{13}{4}$,

and $2y = 4$, or $-\dfrac{5}{2}$;

and $\therefore y = 2$, or $-\dfrac{5}{4}$;

whence $x = \dfrac{120}{y+2} = 30$, or 160,

and the sums are 120, 60, and 30 pounds.

---

17. The sum of three numbers in geometrical progression is 35; and the mean term is to the difference of the extremes as 2 to 3. Required the numbers.

Let $\dfrac{x}{y}$, x, and xy, be the three numbers;

$$\therefore \frac{x}{y} + x + xy = 35,$$

and $x : xy - \frac{x}{y} :: 2 : 3,$

or $1 : y - \frac{1}{y} :: 2 : 3;$

$\therefore\ 2y - \frac{2}{y} = 3,$

and $y^2 - 1 = \frac{3}{2} \cdot y;$

by transposition, $y^2 - \frac{3}{2} \cdot y = 1;$

completing the square, $y^2 - \frac{3}{2} y + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16};$

extracting the root, $y - \frac{3}{4} = \pm \frac{5}{4};$

$\therefore y = 2$, or $-\frac{1}{2}$, which last does not answer the conditions;

$$\therefore \left(\frac{x}{2} + x + 2x = \right) \frac{7x}{2} = 35;$$

$$\therefore x = 10,$$

and the numbers are 5, 10, and 20.

---

18. There are three numbers in geometrical progression, the greatest of which exceeds the least by 15. Also the difference of the squares of the greatest and least is to the sum of the squares of all the three numbers as 5 : 7. Required the numbers.

Let x, xy, xy^2 be the numbers;

then $xy^2 - x = 15,$

and $x^2y^4 - x^2 : x^2y^4 + x^2y^2 + x^2 :: 5 : 7,$

or $y^4 - 1 : y^4 + y^2 + 1 :: 5 : 7;$

$$\therefore\ y^4 - 1 : y^2 + 2 :: 5 : 2,$$

$$\text{and } y^4 - 1 = \frac{5y^2}{2} + 5;$$

$$\text{by transposition, } y^4 - \frac{5}{2} \cdot y^2 = 6;$$

$$\text{completing the square, } y^4 - \frac{5}{2} \cdot y^2 + \frac{25}{16} = 6 + \frac{25}{16} = \frac{121}{16};$$

$$\text{extracting the root, } y^2 - \frac{5}{4} = \pm \frac{11}{4};$$

$$\therefore\ y^2 = 4, \text{ or } -\frac{3}{2}, \text{ which last is impossible,}$$

$$\text{and } y = \pm 2;$$

$$\therefore \text{ from the first equation, } (4x - x =)\ 3x = 15,$$

$$\text{and } x = 5;$$

$\therefore$ the numbers are 5, 10, and 20.

19. The sum of three numbers in geometrical progression is 13, and the product of the mean and the sum of the extremes is 30. Required the numbers.

$$\text{Let the numbers be } \frac{x}{y},\ x, \text{ and } xy;$$

$$\text{then } \frac{x}{y} + x + xy = 13,$$

$$\text{and } \left(\frac{x}{y} + xy\right) \cdot x = 30;$$

$$\therefore \text{ by transposition, } 13 - x = \frac{x}{y} + xy = \frac{30}{x},$$

$$\text{and } 13x - x^2 = 30,$$

$$\text{or } x^2 - 13x = -30;$$

$$\text{completing the square, } x^2 - 13x + \frac{169}{4} = \frac{169}{4} - 30 = \frac{49}{4};$$

$$\text{extracting the root, } x - \frac{13}{2} = \pm \frac{7}{2},$$

and $\therefore\ x = 10$, or 3.

If $x = 3$; then $\frac{3}{y} + 3y = 13 - 3 = 10$,

or $3 + 3y^2 = 10y$;

by transposition, $3y^2 - 10y = -3$,

or $y^2 - \frac{10}{3} \cdot y = -1$;

completing the square, $y^2 - \frac{10}{3} \cdot y + \frac{25}{9} = \frac{25}{9} - 1 = \frac{16}{9}$;

extracting the root, $y - \frac{5}{3} = \pm \frac{4}{3}$;

$\therefore\ y = 3$, or $\frac{1}{3}$,

and the numbers are 1, 3, 9.

If the other value of x be taken, the corresponding values of y are impossible.

---

20. There are three numbers in arithmetical progression, and the square of the first added to the product of the other two is 16; the square of the second added to the product of the other two is 14. What are the numbers?

Let $x-y$, x, $x+y$, be the numbers;

then $2x^2 - xy + y^2 = 16$,

and $2x^2 - y^2 = 14$;

$\therefore$ by subtraction, $2y^2 - xy = 2$,

and by addition, $4x^2 - xy = 30$,

or $2y^2 = 2 + xy$,

and $4x^2 = 30 + xy$;

$\therefore$ by multiplication, $8x^2y^2 = 60 + 32xy + x^2y^2$;

by transposition, $7x^2y^2 - 32xy = 60$,

$$\text{or } x^2y^2 - \frac{32}{7} \cdot xy = \frac{60}{7};$$

completing the square, $x^2y^2 - \frac{32}{7} \cdot xy + \frac{256}{49} = \frac{60}{7}$

$$+ \frac{256}{49} = \frac{676}{49};$$

$$\text{extracting the root, } xy - \frac{16}{7} = \pm \frac{26}{7};$$

$$\therefore xy = 6, \text{ or } -\frac{10}{7};$$

$$\therefore 2y^2 = 2 + xy = 8,$$

$$\text{and } y^2 = 4;$$

$$\therefore y = \pm 2,$$

$$\text{and } 4x^2 = 30 + xy = 36;$$

$$\therefore 2x = \pm 6,$$

$$\text{and } x = \pm 3;$$

$\therefore$ the numbers are 1, 3, 5; or -5, -3, -1. The other value of xy was introduced in the operation, and does not answer the conditions of the question.

---

21. The sum of four whole numbers in arithmetical progression is 20, and the sum of their reciprocals is $\frac{25}{24}$. Required the numbers.

Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, be the numbers;

$$\text{then } 4x = 20,$$

$$\text{or } x = 5.$$

$$\text{Again, } \frac{1}{x-3y} + \frac{1}{x-y} + \frac{1}{x+y} + \frac{1}{x+3y} = \frac{25}{24},$$

$$\text{or } \frac{4x^3 - 20xy^2}{x^4 - 10x^2y^2 + 9y^4} = \frac{25}{24};$$

$$\therefore 25 \times (9y^4 - 250y^2 + 625) = 24 \times (500 - 100y^2),$$

or $9y^4 - 250y^2 + 625 = 24 \times (20 - 4y^2)$;

by transposition, $9y^4 - 154y^2 = -145$;

completing the square, $9y^4 - 154y^2 + \frac{5929}{9} = \frac{5929}{9}$

$-145 = \frac{4624}{9}$;

extracting the root, $3y^2 - \frac{77}{3} = \pm\frac{68}{3}$,

and $3y^2 = 3$, or $\frac{145}{3}$;

$\therefore\ y^2 = 1$, or $\frac{145}{9}$,

and $y = \pm 1$, or $\frac{\pm\sqrt{145}}{3}$,

and $\therefore$ the numbers are 2, 4, 6, 8.

---

22. There is a number consisting of 3 digits, the first of which is to the second as the second to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits will be inverted. Required the number.

Let the digits be represented by x, xy, xy^2;

then $100x + 10xy + xy^2 : x + xy + xy^2 :: 124 : 7$,

or $100 + 10y + y^2 : 1 + y + y^2 :: 124 : 7$;

div^{do}. $99 + 9y : 1 + y + y^2 :: 117 : 7$,

or $11 + y : 1 + y + y^2 :: 13 : 7$;

$\therefore\ 13y^2 + 13y + 13 = 7y + 77$;

by transposition, $13y^2 + 6y = 64$,

or $y^2 + \frac{6}{13}\cdot y = \frac{64}{13}$;

completing the square, $y^2 + \frac{6}{13} \cdot y + \overline{\frac{3}{13}}|^2 = \frac{64}{13} + \frac{9}{\overline{13}|^2} = \frac{841}{169}$;

extracting the root, $y + \frac{3}{13} = \pm \frac{29}{13}$;

$\therefore \; y = 2, \text{ or } -\frac{32}{13}$;

also $100x + 10xy + xy^2 + 594 = 100xy^2 + 10xy + x$;

by transposition, $99x + 594 = 99xy^2$,

or $x + 6 = xy^2 = 4x$;

$\therefore$ by transposition, $6 = 3x$,

and $2 = x$;

$\therefore$ the digits are 2, 4, 8, and the number is 248.

---

23. There are five whole numbers, the three first of which are in geometric progression; the three last in arithmetic progression, the second number being the second difference. The sum of the four last = 40, and the product of the second and last = 64. Required the numbers.

Let x = the first,

and y = the common ratio of the three first;

$\therefore$ the numbers are $x, xy, xy^2, xy^2 + xy, xy^2 + 2xy$;

$\therefore \; 3xy^2 + 4xy = 40$,

and $x^2y^3 + 2x^2y^2 = 64$.

Multiplying the first equation by xy, and the second by 3,

$3x^2y^3 + 4x^2y^2 = 40xy$,

and $3x^2y^3 + 6x^2y^2 = 192$;

$\therefore$ by subtraction, $2x^2y^2 = 192 - 40xy$;

by transposition, $2x^2y^2 + 40xy = 192$,

or $x^2y^2 + 20xy = 96$;

completing the square, $x^2y^2 + 20xy + 100 = 196$;

extracting the root, $xy + 10 = \pm 14$;

$$\therefore\ xy = 4,\ \text{or}\ -24.$$

Now from the first equation, $xy\,.\,(3y + 4) = 40$,

$$\text{or}\ 4\,.\,(3y + 4) = 40;$$

$$\therefore\ 3y + 4 = 10;$$

by transposition, $3y = 6$,

$$\text{and}\ y = 2;$$

$$\therefore\ \text{also}\ x = 2,$$

and the numbers are 2, 4, 8, 12, 16.

---

24. There are two casks *A* and *B*; of which, *A* the greater holds 312 gallons. Into *A* a certain quantity of wine is put, and *B* is filled with water; then water is conveyed out of *B* into *A* in the following manner. First, a number of gallons is taken, which is less by two than the square root of the number of gallons in *A*, then a quantity less than the former by two gallons, and so on. Now when *B* is in this manner exactly emptied, *A* is exactly full: and it is known that 8 gallons were taken out of *B* at one time, after which the quantity left in *B* was 12 gallons. Required the number of gallons of wine in *A*.

Since the quantities taken out of *B* are in a decreasing progression, whose common difference is 2, and one term of this progression is 8, therefore the next terms are 6, 4, 2, the sum of which is = 12; and therefore the quantity last drawn out is 2 gallons. Let x^2 = the number of gallons of wine in *A*, then $x - 2$, $x - 4$, &c. are the numbers of gallons drawn each successive time; and the number of terms is evidently $\frac{x}{2} - 1$; and therefore the whole quantity drawn from *B* is $\frac{x}{2}\,.\left(\frac{x}{2} - 1\right) = \frac{x^2}{4} - \frac{x}{2}$.

$$\therefore\ x^2 + \frac{x^2}{4} - \frac{x}{2} = 312.$$

$$\text{or } \frac{5x^2}{4} - \frac{x}{2} = 312;$$

$$\therefore\ x^2 - \frac{2}{5}x = \frac{1248}{5};$$

completing the square, $x^2 - \frac{2}{5}x + \frac{1}{25} = \frac{1248}{5} + \frac{1}{25}$

$$= \frac{6241}{25};$$

extracting the root, $x - \frac{1}{5} = \pm\frac{79}{5}$,

and $x = 16$, or $-\frac{78}{5}$, which last will not answer the conditions; therefore $x^2 = 256 =$ the number required.

25. The diagonals of 4 squares are in an increasing geometrical progression, and the product of the squares of the diagonals of the extremes is to the product of the diagonals of the means as a side of the third is to the square root of the common ratio divided by $4\sqrt{2}$. Required the diagonal of the third square, and the common ratio, supposing their difference equal to 45.

Let $\frac{x}{y}$, x, xy, and $xy^2 =$ the diagonals;

then since the diagonal : a side :: $\sqrt{2} : 1$;

the side of the third $= \frac{xy}{\sqrt{2}}$,

$$\text{and } \frac{x^2}{y^2} \times x^2y^4 : x \times xy :: \frac{xy}{\sqrt{2}} : \frac{\sqrt{y}}{4\sqrt{2}};$$

$$\therefore\ x^2y : 1 :: 4x\sqrt{y} : 1;$$

$$\therefore\ x^2y = 4x\sqrt{y},$$

$$\text{and } xy^{\frac{1}{2}} = 4.$$

$$\text{Now } y - xy = 45,$$

$$\text{or } y - 4y^{\frac{1}{2}} = 45;$$

$$\text{completing the square, } y - 4y^{\frac{1}{2}} + 4 = 49;$$

$$\text{extracting the root, } y^{\frac{1}{2}} - 2 = \pm 7;$$

$\therefore y^{\frac{1}{2}} = 9$, or -5, which last does not agree with the conditions; and $\therefore y = 81$;

$$\text{whence } x = \frac{4}{y^{\frac{1}{2}}} = \frac{4}{9};$$

and $\therefore$ the diagonal of the third square $= 36$.

---

26. Two persons, *A* and *B*, traded together. *A* gained every year £.3 more than the preceding year, and the last year he gained £.17. His whole gain was £.57. *B*, in the four first years gained £.52, and if what *A* put into the common stock be added to what *B* gained the second year, the sum will be £.13. How many years did they remain in trade, and what were their original stocks?

Since *A*'s annual gains are in an increasing arithmetical progression, whose common difference is 3; the last term 17, and sum 57, if $n =$ the number of terms, then (*Alg.* 212.),

$$(34 - (n-1) . 3) . \frac{n}{2} = 57,$$

$$\text{or } 37n - 3n^2 = 114;$$

$$\therefore n^2 - \frac{37}{3}n = -\frac{114}{3};$$

completing the square, $n^2 - \frac{37}{3}n + \overline{\frac{37}{6}}|^2 = \frac{1369}{36} - \frac{114}{3}$

$$= \frac{1}{36};$$

$$\text{extracting the root, } n - \frac{37}{6} = \pm \frac{1}{6};$$

$$\therefore n = 6, \text{ or } \frac{19}{3};$$

hence they remained 6 years in trade, and consequently *A*'s gain the first year was £.2, and his gain in four years was £.26.

$$\text{Let } \therefore x = A\text{'s stock},$$

$$\text{and } 26 : 52 :: x : B\text{'s stock} = 2x,$$

$$\text{and } A\text{'s gain the second year being £.5},$$

$$x : 2x :: 5 : B\text{'s gain} = 10;$$

$$\therefore x + 10 = 13,$$

$$\text{and } x = 3;$$

$$\therefore A\text{'s stock was £.3, and } B\text{'s £.6.}$$

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27. A pyramidical pile of cannon-balls, the base of which was an equilateral triangle, was all used in an engagement, except the three lowest layers, and 4 balls of the next layer; these were afterwards formed into a pile with a rectangular base, having as many balls in one side of the lowest layer, as there were in a side of the lowest layer of the pyramidical pile, and 4 in the adjacent side. What was the number of balls; and what the number of layers in each pile when complete?

Let x = the number of balls in a side of the lowest layer;

$$\therefore x \cdot \frac{x+1}{2} = \text{the number of balls in that layer,}$$

$$\text{and } (x-1) \cdot \frac{x}{2} = \text{the number in the next,}$$

$$\text{and } (x-2) \cdot \frac{x-1}{2} = \text{the number in the third;}$$

$$\therefore \frac{3x^2 - 3x + 2}{2} + 4 = \text{the whole number left.}$$

Now since there were only 4 balls in one side of the second pile, there can only be four layers, which will contain $4x$, $3 \cdot (x - 1)$, $2 \cdot (x - 2)$, and $x - 3$ balls respectively;

$$\therefore \frac{3x^2 - 3x + 2}{2} + 4 = 10x - 10,$$

$$\text{or } 3x^2 - 3x + 2 + 8 = 20x - 20;$$

$$\text{by transposition, } 3x^2 - 23x = -30,$$

$$\text{or } x^2 - \frac{23}{3}x = -10;$$

completing the square, $x^2 - \frac{23}{3} \cdot x + \left.\frac{23}{6}\right|^2 = \frac{529}{36} - 10$ $= \frac{169}{36}$;

$$\text{extracting the root, } x - \frac{23}{6} = \pm \frac{13}{6},$$

and $x = 6$, or $\frac{5}{3}$, which last cannot answer the conditions of the problem. Hence there were 6 layers in the first pile, and they contained 1, 3, 6, 10, 15, 21 balls, respectively; $\therefore$ the whole number of balls in the first pile was 56, and in the second 50.

(31.) In the preceding solutions it may be observed, that, in many instances, values of the unknown quantities are deduced, which do not agree with the conditions of the problems. This is always the case when the roots of the equations are negative; and the circumstance arises from that peculiar quality of an algebraic expression, by which it is denominated either positive or negative. The product of two or any even number of such quantities, whether all of them are positive or all negative, will only be affected with a positive sign: thus

the quantity xy will represent the product of $+x \times +y$, or of $-x \times -y$; and a^2, of $+a \times +a$, or of $-a \times -a$: consequently, in the reduction of such quantities to their constituent factors by the rules of division or evolution, these factors may be considered either as all positive or all negative. But in common language, in which the conditions of a problem are expressed, quantity or number is from its very nature what in Algebra is meant by the term positive, *i. e.* it increases any homogeneous quantity to which it is added, and diminishes any one from which it is subtracted. Hence it may be understood, why, when quadratic equations are formed to express the conditions of a problem, the resulting roots may exceed in number what appear to be required as answers to the problem, and why such as are negative cannot be applied to its conditions.

These roots or values, however, though inapplicable in their present shape, will, if assumed as positive, become correct answers to the problem under a different modification of the conditions. In the equations thence deduced, these former negative values will appear as positive roots, and the former positive values as negative roots. Thus, if Prob. 11, page 192, be transformed into the following, "A detachment from an army was marching "in regular column with 5 *fewer* in depth than in front; "but upon the enemy coming in sight the front was in- "creased till it became = 845 − the original front; "and by this movement the detachment was drawn up "in 5 lines. Required the number of men;" from the solution of this problem the number is found to be 3900, answering to the number which would be found from using the negative value of x in the original problem; and the equation for determining this ($x^2 - 5x = 4225 - 5x$) differs from the other only in the sign of x.

In Prob. 18, page 196, x is found to be equal to ± 4, where the negative value shews that if the trading vessel had turned out of its first course in a direction contrary to CE, or on the opposite side of the line AC, it would have been taken after sailing 4 miles in that direction.

In Prob. 1, page 206, the negative value of x is found to be -130. But if the problem be modified so as to become "A merchant sold a quantity of brandy, by "which he lost £.39 *more* than the prime cost, and found "that his loss was as much *per* cent. as the brandy cost "him. What was that price?" the equation for determining the price, is $\frac{x^2}{100} = 39 + x$, which is deduced from the equation to the original problem by changing the sign of x, the positive value of which is in this case 130.

Also, in Prob. 4, page 207, the negative value of x is $-\frac{15}{2}$. Now if the problem were "Bought two sorts of "linen, for the finer of which I gave 6 crowns *more than* "*for the other*. An ell of the finer cost as many shillings "as there were ells of the finer. Also 28 ells of the coarser "(which was the whole quantity) sold at such a price, "that 8 ells cost as many shillings as one ell of the finer. "How many ells were there of the finer; and what was "the value of each piece?" an equation arises differing from the equation to the original problem only in the sign of x, and whose positive root is $\frac{15}{2}$; whence there were $7\frac{1}{2}$ ells of the finer at 7*s.* 6*d.* *per* ell, the whole price of which was therefore £.2. 16*s.* 3*d.*, and the whole price of the coarser was £.1. 6*s.* 3*d.* And in the very same manner, all the other problems may be transformed.

(32.) The same reasoning will apply to the case i which all the roots of the resulting equation are negative None of its values can in this case be applied to satisf the conditions of the problem; but if the conditions ar properly modified, equations may be deduced, of whic these values rendered positive will become roots, and wi satisfy such conditions.

The same observation holds, if the resulting values b the square roots of negative quantities, with this exceptior that such roots can never be applied to satisfy the condi tions of the problem under any modification whatever.

Sect. XI.

PRAXIS.

I. *Simple Equations involving only one unknown Quantity.*

1. Given $36 - 4x = 17x + 15$, to find the value of x.
Answer $x = 1$.

2. Given $2x + 7 - \frac{x}{2} = 28 - 2x$, to find the value of x.

Ans. $x = 6$.

× 3. Given $x^2 + 17x = 37x - 4x^2$, to find the value of x.
Ans. $x = 4$.

4. Given $\frac{x}{6} - \frac{x}{4} + 10 = \frac{x}{3} - \frac{x}{2} + 11$, to find the value of x.

Ans. $x = 12$.

5. Given $\frac{x+1}{3} + 1 = \frac{3x-4}{5}$, to find the value of x.

Ans. $x = 8$.

6. Given $\frac{7x+3}{4} + 5x = 20 + \frac{4x-5}{7}$, to find the value of x.

Ans. $x = 3$.

7. Given $\frac{5x+4}{3} + 8x = \frac{16-x}{3} + 36$, to find the value of x. Ans. $x = 4$.

8. Given $\frac{7-x}{2} + 4 = \frac{3x-11}{4} + \frac{8x+15}{6}$, to find the value of x. Ans. $x = 3$.

9. Given $\frac{2x-3}{18} + \frac{18-x}{3} = \frac{11x-3}{9} - \frac{5}{2}$, to find the value of x. Ans. $x = 6$.

10. Given $\frac{3x+4}{7} - \frac{19+x}{3} = 35 - 5x$, to find the value of x.

Ans. $x = 8$.

11. Given $4x - \frac{17+2x}{3} = 7 - \frac{7x+10}{4}$, to find the value of x.

Ans. $x = 2$.

12. Given $\frac{21-3x}{3} - \frac{4x+6}{9} = 6 - \frac{5x+1}{4}$, to find the value of x.

Ans. $x = 3$.

13. Given $\frac{8x+4}{11} - \frac{7x+3}{2} = \frac{32-4x}{3} - \frac{5x+13}{2}$, to find the value of x.

Ans. $x = 5$.

14. Given $x + \frac{27-9x}{4} - \frac{5x+2}{6} = \frac{61}{12} - \frac{2x+5}{3} - \frac{29+4x}{12}$, to find the value of x.

Ans. $x = 5$.

15. Given $\frac{7x+3}{11}+\frac{15x+8}{13}=x+\frac{25-x}{2}$, to find the value of x.

Ans. $x = 9$.

16. Given $\frac{4x-6}{5}-\frac{3x-4}{3}=5\tfrac{1}{3}-\frac{27+4x}{9}$, to find the value of x.

Ans. $x = 9$.

17. Given $2x-\frac{4x-6}{11}=\frac{2x+11}{5}-\frac{7-8x}{7}$, to find the value of x.

Ans. $x = 7$.

18. Given $\frac{7x+8}{4}-\frac{3x+1}{5}=\frac{9x-12}{8}-\frac{29-8x}{10}$, to find the value of x.

Ans. $x = 8$.

19. Given $4-\frac{3x+15}{4}-\frac{7-6x}{9}=\frac{35}{36}+x-\frac{9x+11}{8}$, to find the value of x.

Ans. $x = 3$.

20. Given $\frac{7x-43}{12}+14\tfrac{1}{2}-\frac{13+5x}{8}=276-\frac{3x-12}{9}-\frac{4x+17}{6}-13x$, to find the value of x.

Ans. $x = 19$.

21. Given $4x+\frac{1}{10}-\frac{3x-13}{16}-\frac{12+7x}{9}=7x-33-\frac{9+5x}{10}-\frac{11x-17}{8}$, to find the value of x.

Ans. $x = 15$.

22. Given $\frac{31+4x}{3} - \frac{5x+13}{8} - \frac{15+x}{16} = 47\frac{3}{4} + \frac{16-10x}{11} - \frac{4x+37}{7}$, to find the value of x.

Ans. $x = 17$.

23. Given $\frac{9x+11}{36} = \frac{3x-16}{5x-9} + \frac{x}{4}$, to find the value of x.

Ans. $x = 9$.

24. Given $\frac{20x+36}{25} + \frac{7x+12}{9x-16} = \frac{4x}{5} + \frac{86}{25}$, to find the value of x.

Ans. $x = 4$.

25. Given $\frac{10x+17}{18} - \frac{12x+2}{11x-8} = \frac{5x-4}{9}$, to find the value of x.

Ans. $x = 4$.

26. Given $\frac{18x-19}{28} + \frac{13x+7}{6x+14} = \frac{9x+15}{14}$, to find the value of x.

Ans. $x = 7$.

27. Given $\frac{10+x}{5} : \frac{3x-5}{7} :: 14 : 5$, to find the value of x.

Ans. $x = 4$.

28. Given $\frac{17-4x}{4} : \frac{18+x}{3} - 2x :: 5 : 4$, to find the value of x.

Ans. $x = 3$.

29. Given $16x+5 : \frac{4x+14}{9x+31} :: 36x+10 : 1$, to find the value of x.

Ans. $x = 5$.

30. Given $\frac{4x+7}{6x-37} : 1 :: 2x+21 : 3x-16$, to find the value of x.

ANS. $x = 7$.

31. Given $5x + \frac{7x+9}{4x+3} = 9 + \frac{10x^2-18}{2x+3}$, to find the value of x.

ANS. $x = 3$.

32. Given $\sqrt[3]{5x+24} - 1 = 3$, to find the value of x.

ANS. $x = 8$.

33. Given $\sqrt[5]{9x-4} + 6 = 8$, to find the value of x.

ANS. $x = 4$.

34. Given $\sqrt{x+9} = 1 + \sqrt{x}$, to find the value of x.

ANS. $x = 16$.

35. Given $\sqrt{x-32} = 16 - \sqrt{x}$, to find the value of x.

ANS. $x = 81$.

36. Given $\sqrt{4x+17} = 2\sqrt{x} + 1$, to find the value of x.

ANS. $x = 16$.

37. Given $\frac{\sqrt{9x}-6}{\sqrt{x}+3} = \frac{5+\sqrt{9x}}{\sqrt{x}+25}$, to find the value of x.

ANS. $x = 9$.

38. Given $\frac{\sqrt{6x}-2}{\sqrt{6x}+2} = \frac{4\sqrt{6x}-9}{4\sqrt{6x}+6}$, to find the value of x.

ANS. $x = 6$.

39. Given $\frac{5x-9}{\sqrt{5x}+3} - 1 = \frac{\sqrt{5x}-3}{2}$, to fin
value of x.

Ans. $x = 5$.

40. Given $\sqrt{1 + x\sqrt{x^2+12}} = 1 + x$, to fin
value of x.

Ans. $x = 2$.

41. Given $\sqrt{x} + \sqrt{16+x} = \frac{40}{\sqrt{16+x}}$, to finc
value of x.

Ans. $x = 9$.

II. *Simple Equations involving two unknown Quant*

1. Given $\left.\begin{array}{r} x + 15y = 36, \\ \text{and } y + 3x = 20, \end{array}\right\}$ to find the values of x an

Ans. $\begin{cases} x = 6, \\ y = 2. \end{cases}$

2. Given $\left.\begin{array}{r} 4x + 9y = 47, \\ \text{and } 8x - 13y = 1, \end{array}\right\}$ to find the values of x an

Ans. $\begin{cases} x = 5, \\ y = 3. \end{cases}$

3. Given $\left.\begin{array}{r} \frac{x}{6} + \frac{y}{4} = 6, \\ \text{and } \frac{x}{4} + \frac{y}{6} = 5\frac{2}{3}; \end{array}\right\}$ to find the values of x and

Ans. $\begin{cases} x = 12, \\ y = 16. \end{cases}$

4. Given $\left.\begin{aligned}\frac{x}{8}+8y&=194,\\ \text{and } \frac{y}{8}+8x&=131,\end{aligned}\right\}$ to find the values of x and y.

ANS. $\begin{cases}x=16,\\ y=24.\end{cases}$

5. Given $\left.\begin{aligned}\frac{3x-1}{4}+2y-5&=8,\\ \text{and } \frac{3y-2}{6}+2x-8&=7\tfrac{2}{3},\end{aligned}\right\}$ to find the values of x and y.

ANS. $\begin{cases}x=7,\\ y=4.\end{cases}$

6. Given $\left.\begin{aligned}9x+\frac{8y}{5}&=70,\\ \text{and } 7y-\frac{13x}{3}&=44,\end{aligned}\right\}$ to find the values of x and y.

ANS. $\begin{cases}x=6,\\ y=10.\end{cases}$

7. Given $\left.\begin{aligned}\frac{7+x}{5}-\frac{2x-y}{4}&=3y-5,\\ \text{and } \frac{5y-7}{2}+\frac{4x-3}{6}&=18-5x,\end{aligned}\right\}$ to find the values of x and y.

ANS. $\begin{cases}x=3,\\ y=2.\end{cases}$

8. Given $\left.\begin{aligned}x+1-\frac{3y+4x}{7}&=7-\frac{9y+33}{14},\\ \text{and } y-3-\frac{5x-4y}{2}&=x-\frac{11y-19}{4},\end{aligned}\right\}$ to find the values of x and y.

ANS. $\begin{cases}x=6,\\ y=5.\end{cases}$

9\. Given $4x + \frac{15 - x}{4} = 2y + 5 + \frac{7x + 11}{16}$, and $3y - \frac{2x + y}{5} = x + \frac{2y + 4}{3}$, to find the values of x and y.

ANS. $\begin{cases} x = 3, \\ y = 4. \end{cases}$

10\. Given $x - \frac{3x + 5y}{17} + 17 = 5y + \frac{4x + 7}{3}$, and $\frac{22 - 6y}{3} - \frac{5x - 7}{11} = \frac{x + 1}{6} - \frac{8y + 5}{18}$, to find the values of x and y.

ANS. $\begin{cases} x = 8, \\ y = 2. \end{cases}$

11\. Given $\frac{5x - 3}{6} + \frac{3y - x}{3} = 4 + \frac{3x - 19}{2}$, and $\frac{2x + y}{2} - \frac{9x - 7}{8} = \frac{3y + 9}{4} - \frac{4x + 5y}{16}$, to find the values of x and y.

Ans. $\begin{cases} x = 9, \\ y = 4. \end{cases}$

12\. Given $\frac{7x + 6}{11} + \frac{4y - 9}{3} = 3x - \frac{13 - x}{2} - \frac{3y - x}{5}$,
and $3x + 4 : 2y - 3 :: 5 : 3$,
to find the values of x and y.

ANS. $\begin{cases} x = 7, \\ y = 9. \end{cases}$

13\. Given $\frac{5x + 13}{2} - \frac{8y - 3x - 5}{6} = 9 + \frac{7x - 3y + 1}{3}$,
and $\frac{x + 7}{3} : \frac{3y - 8}{4} + 4x :: 4 : 21$,
to find the values of x and y.

ANS. $\begin{cases} x = 5, \\ y = 4. \end{cases}$

14. Given $\frac{3x+4y+3}{10} - \frac{2x+7-y}{15} = 5 + \frac{y-8}{5}$,

and $\frac{9y+5x-8}{12} - \frac{x+y}{4} = \frac{7x+6}{11}$,

to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 9. \end{cases}$

15. Given $13x + \frac{4y-17+x}{12} - \frac{15-3x}{4} = \frac{12y+11}{3}$

$- \frac{12x+7y+28}{6}$,

and $\frac{9x+18}{4} - \frac{12+5y-6x}{5} = \frac{15x-3y-5}{8}$

$- \frac{7x+y-10}{15}$,

to find the values of x and y.

Ans. $\begin{cases} x = 2, \\ y = 11. \end{cases}$

16. Given $\frac{6x+9}{4} + \frac{3x+5y}{4x-6} = 3\frac{1}{4} + \frac{3x+4}{2}$,

and $\frac{8y+7}{10} + \frac{6x-3y}{2y-8} = 4 + \frac{4y-9}{5}$,

to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 9. \end{cases}$

17. Given $4x - 34\frac{1}{3} - \frac{4y+13x}{27-6y} = \frac{12x+8}{3}$,

and $3x + \frac{21-4y}{4x-10} = \frac{18x+13}{6} - 2\frac{1}{9}$,

to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 5. \end{cases}$

18. Given $4x+3y+\dfrac{24+\dfrac{11y}{2}}{2x+1}=\dfrac{16x^2+12xy-8x+5y+28}{4x-2}$,

and $2x+4=3y+\dfrac{8x^3-18y^2+108}{4x+6y+3}$,

to find the values of x and y.

ANS. $\begin{cases} x=3, \\ y=2. \end{cases}$

19. Given $\dfrac{\dfrac{7x}{4}+6y}{5}-\dfrac{\dfrac{3y+6}{5}-\dfrac{3x-2}{10}}{8}=5-\dfrac{x}{16}$,

and $\dfrac{3x}{2}+\dfrac{2y}{3}+2\frac{1}{2}:\dfrac{x}{2}-\dfrac{y}{3}+\dfrac{1}{6}::10\frac{1}{2}:1\frac{1}{6}$,

to find the values of x and y.

ANS. $\begin{cases} x=4, \\ y=3. \end{cases}$

20. Given $4+\dfrac{12y-\dfrac{6y+2}{x}}{11}=y+\dfrac{\dfrac{3xy-31}{11}+10x+13}{3x}$,

and $\dfrac{2x}{3}-\dfrac{3x-5}{y+7}=\dfrac{4xy+\dfrac{170}{3}}{6y+27}$,

to find the values of x and y.

ANS. $\begin{cases} x=7, \\ y=2. \end{cases}$

III. *Pure Quadratics and others which may be solved without completing the square.*

1. Given $5x^2-16=20+x^2$, to find the values of x.

ANS. $x=\pm 3$.

2. Given $x+y : y :: 3 : 1,$ and $xy = 8,$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 4 \\ y = \pm 2. \end{cases}$

3. Given $x - y : y :: 3 : 5,$ and $x^2 + 4y^2 = 164,$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 8, \\ y = \pm 5. \end{cases}$

4. Given $x^2+y^2 : x^2-y^2 :: 25 : 7,$ and $xy^2 = 36,$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \\ y = 3. \end{cases}$

5. Given $x^2 - xy = 48,$ and $xy - y^2 = 12,$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 8, \\ y = \pm 2. \end{cases}$

6. Given $x+y : x^2-y^2 :: 1 : 3,$ and $xy = 10,$ to find the values of of x and y.

Ans. $\begin{cases} x = 5, \text{ or } - 2, \\ y = 2, \text{ or } - 5. \end{cases}$

7. Given $x^3+y^3 : x^3-y^3 :: 234 : 109,$ and $xy^2 = 175,$ to find the values of x and y.

Ans. $\begin{cases} x = 7, \\ y = 5. \end{cases}$

8. Given $x^2 - xy : xy - y^2 :: 5 : 3,$ and $x^2y = 75,$ to find the values of x and y.

Ans. $\begin{cases} x = 5, \\ y = 3. \end{cases}$

16. Given $\dfrac{1}{1-\sqrt{1-x^2}} - \dfrac{1}{1+\sqrt{1-x^2}} = \dfrac{\sqrt{3}}{x^2}$, to find the values of x.

Ans. $x = \pm \dfrac{1}{2}$.

17. Given $\left.\begin{aligned} x^2y + y^2 &= 116, \\ \text{and } xy^{\frac{1}{2}} + y &= 14, \end{aligned}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } 2\sqrt{\dfrac{2}{5}}, \\ y = 4, \text{ or } 10. \end{cases}$

18. Given $\left.\begin{aligned} \sqrt[3]{x} + \sqrt[3]{y} &= 6, \\ \text{and } x + y &= 72, \end{aligned}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 64, \text{ or } 8, \\ y = 8, \text{ or } 64. \end{cases}$

19. Given $\left.\begin{aligned} 4x^2 + \frac{5}{2} &= \frac{x^2}{y} + 10y, \\ \text{and } x^2 + 3y &= 55, \end{aligned}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 5, \\ y = 10. \end{cases}$

20. Given $\dfrac{1}{x+\sqrt{2-x^2}} + \dfrac{1}{x-\sqrt{2-x^2}} = ax$, to find the values of x.

Ans. $x = \pm \sqrt{\dfrac{a+1}{a}}$.

21. Given $\left.\begin{aligned} x^{\frac{4}{3}} + y^{\frac{2}{5}} &= 20, \\ \text{and } x^{\frac{2}{3}} + y^{\frac{1}{5}} &= 6, \end{aligned}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 8, \text{ or } \pm \sqrt{8}, \\ y = 32, \text{ or } 1024. \end{cases}$

22. Given $x^4+2x^2y^2+y^4 = 1296 - 4xy\,.\,(x^2+xy+y^2)$,
and $x - y = 4$,
to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } -1, \\ y = 1, \text{ or } -5. \end{cases}$

23. Given $\left.\begin{array}{l} \dfrac{\sqrt{x}+\sqrt{x-y}}{\sqrt{x}-\sqrt{x-y}} = 4, \\ \text{and } \sqrt{x} : \sqrt{y} :: \sqrt{y} : 4, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \dfrac{625}{16}, \\ y = 25. \end{cases}$

24. Given $\left.\begin{array}{l} \dfrac{1}{y} - \dfrac{1}{x} = \dfrac{1}{4}, \\ \text{and } x^2y - xy^2 = 16, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \\ y = 2. \end{cases}$

25. Given $\left.\dfrac{\sqrt{4x+1}+\sqrt{4x}}{\sqrt{4x+1}-\sqrt{4x}} = 9,\right\}$ to find the value of x.

Ans. $x = \dfrac{4}{9}$.

26. Given $\left.\begin{array}{l} x^4y^3 - x^3y^4 = 216, \\ \text{and } x^2y - xy^2 = 6, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = 3, \\ y = 2. \end{cases}$

27. Given $\left.\begin{array}{l} x^2 + x\sqrt[3]{xy^2} = 208, \\ \text{and } y^2 + y\sqrt[3]{x^2y} = 1053, \end{array}\right\}$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 8, \\ y = \pm 27. \end{cases}$

28. Given $x^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} + y^{\frac{3}{2}} = 1009,$ and $x^3 + x^{\frac{3}{2}}y^{\frac{3}{2}} + y^3 = 582193,$ to find the values of x and y.

Ans. $\begin{cases} x = 81, \text{ or } 16, \\ y = 16, \text{ or } 81. \end{cases}$

29. Given $x^2 + y^2 + xy \cdot (x + y) = 68,$ and $x^3 + y^3 - 3x^2 = 12 + 3y^2,$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \text{ or } 2, \\ y = 2, \text{ or } 4. \end{cases}$

30. Given $xy \cdot (x + y) = 84,$ and $x^2y^2 \cdot (x^2 + y^2) = 3600,$ to find the values of x and y.

Ans. $\begin{cases} x = 4, \text{ or } 3, \\ y = 3, \text{ or } 4. \end{cases}$

31. Given $\dfrac{x^2 + xy + y^2}{x + y} = 7,$ and $\dfrac{x^2 - xy + y^2}{x - y} = 9,$ to find the values of x and y.

Ans. $\begin{cases} x = \pm 6, \\ y = 3. \end{cases}$

IV. *Adfected Quadratics involving only one unknown Quantity.*

1. Given $x^2 + 8x = 33$, to find the values of x.

Ans. $x = 3$, or $- 11$.

2. Given $x^2 + 6x + 4 = 59$, to find the values of x.

Ans. $x = 5$, or $- 11$.

3. Given $x^2 - 10x + 17 = 1$, to find the values of x.

Ans. $x = 8$, or 2.

4 Given $x^2 - x + 3 = 45$, to find the values of x.

Ans. $x = 7$, or $- 6$.

5. Given $5x^2 - 4x + 3 = 159$, to find the values of x.

Ans. $x = 6$, or $-\frac{26}{5}$.

6. Given $7x^2 - 21x + 13 = 293$, to find the values of x.

Ans. $x = 8$, or -5.

7. Given $\frac{x^2}{3} + \frac{4x}{5} - 19 = 15\frac{1}{5}$, to find the values of x.

Ans. $x = 9$, or $-\frac{57}{5}$.

8. Given $x + 4 + \frac{7x - 8}{x} = 13$, to find the values of x.

Ans. $x = 4$, or -2.

9. Given $6x + \frac{35 - 3x}{x} = 44$, to find the values of x.

Ans. $x = 7$, or $-\frac{5}{6}$.

10. Given $14 + 4x - \frac{7 + x}{x - 7} = 3x + \frac{9 + 4x}{3}$, to find the values of x.

Ans. $x = 9$, or 28.

11. Given $\frac{3x + 4}{5} - \frac{30 - 2x}{x - 6} = \frac{7x - 14}{10}$, to find the values of x.

Ans. $x = 36$, or 12.

12. Given $\frac{15 - x}{4} - \frac{12 - 3x}{4x - 5} = 7x - \frac{23x + 60}{7}$, to find the values of x.

Ans. $x = 3$, or $-\frac{229}{148}$.

13. Given $\frac{x+11}{x} + \frac{9+4x}{x^2} = 7$, to find the values of x.

ANS. $x = 3$, or $-\frac{1}{2}$.

14. Given $\frac{2x+9}{9} + \frac{4x-3}{4x+3} = 3 + \frac{3x-16}{18}$, to find the values of x.

ANS. $x = 6$, or $-\frac{19}{4}$.

15. Given $\frac{3x-7}{x} + \frac{4x-10}{x+5} = 3\frac{1}{2}$, to find the values of x.

ANS. $x = 7$, or $-\frac{10}{7}$

X 16. Given $\frac{5x-12}{9} + \frac{3x-24}{4x-12} = 9 - \frac{7x-34}{15}$, to find the values of x.

ANS. $x = 12$, or $\frac{477}{184}$.

17. Given $\frac{4x-5}{x} - \frac{3x-7}{3x+7} = \frac{9x+23}{13x}$, to find the values of x.

ANS. $x = 2$, or $-\frac{154}{45}$.

18. Given $2x + 18 - \frac{8x^2+16}{4x+7} = 27 - \frac{12x-11}{2x-3}$, to find the values of x.

ANS. $x = 8$, or 5.

19. Given $\frac{x+4}{x+6}+\frac{5}{2x+4}=\frac{3x+7}{3x+4}$, to find the values of x.

ANS. $x=8$, or $-\frac{2}{3}$.

20. Given $\frac{4}{2x+3}+\frac{3x+6}{5x+18}=\frac{3x+5}{5x}$, to find the values of x.

ANS. $x=6$, or $-\frac{45}{2}$.

21. Given $\frac{8}{9+5x}+\frac{8x-17}{2+4x}=\frac{4x+3}{2x+12}$, to find the values of x.

ANS. $x=3$, or $-\frac{283}{137}$.

22. Given $\frac{12}{5-x}+\frac{8}{4-x}=\frac{32}{x+2}$, to find the values of x.

ANS. $x=2$, or $\frac{58}{13}$.

23. Given $\frac{3}{6x-x^2}+\frac{6}{x^2+2x}=\frac{11}{5x}$, to find the values of x.

ANS. $x=3$, or $\frac{26}{11}$.

24. Given $\frac{4x^2+7x}{19}+\frac{5x-x^2}{3+x}=\frac{4x^3}{9}$, to find the values of x.

ANS. $x=3$, or $-\frac{87}{10}$.

25. Given $\frac{x^4+2x^3+8}{x^2+x-6} = x^2+x+8$, to find the values of x.

Ans. $x = 4$, or $-\frac{14}{3}$.

26. Given $\frac{x+12}{x} + \frac{x}{x+12} = 5\frac{3}{15}$, to find the values of x.

Ans. $x = 3$, or -15.

27. Given $\sqrt{4x+5} \times \sqrt{7x+1} = 30$, to find the values of x.

Ans. $x = 5$, or $-\frac{179}{28}$.

28. Given $\frac{\sqrt{x}+9}{\sqrt{x}} = \frac{\sqrt{9x} - 3\frac{4}{5}}{9-\sqrt{x}}$, to find the values of x.

Ans. $x = 25$, or $\frac{5041}{400}$.

29. Given $\sqrt{x^5} - \frac{40}{\sqrt{x}} = 3x$, to find the values of x.

Ans. $x = 4$, or $\overline{-5}|^{\frac{2}{3}}$.

30. Given $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 44$, to find the values of x.

Ans. $x = \pm 8$, or $\pm\overline{-11}|^{\frac{3}{2}}$.

31. Given $4x^{\frac{1}{3}} + x^{\frac{1}{6}} = 39$, to find the values of x.

Ans. $x = 729$, or $\overline{\frac{13}{4}}|^{6}$.

32. Given $\frac{8}{x^3} + 2 = \frac{17}{x^{\frac{3}{2}}}$, to find the values of x.

Ans. $x = 4$, or $\sqrt[3]{\frac{1}{4}}$.

33. Given $3\,x^n \sqrt[3]{x^n} - \frac{4\,x^n}{\sqrt[3]{x^n}} = 4$, to find the values of x.

Ans. $x = (8)^{\frac{1}{2n}}$, or $\left(-\frac{8}{27}\right)^{\frac{1}{2n}}$.

34. Given

$$\frac{16-4\sqrt{x}}{8-3\sqrt{x}} = \frac{88+33\sqrt{x}}{4+\sqrt{x}} + \frac{x^2-5x+11}{(8-3\sqrt{x}).(4+\sqrt{x})},$$

to find the values of x.

Ans. $x = 93$, or 7.

35. Given

$$\frac{54-9\sqrt{x}}{x+2\sqrt{x}} = \frac{23x-46\sqrt{x}}{6+\sqrt{x}} + \frac{7x^2-3x+4}{(x+2\sqrt{x})\times(6+\sqrt{x})},$$

to find the values of x.

Ans. $x = 5$, or $-\frac{32}{15}$.

36. Given $x+\sqrt{x} : x-\sqrt{x} :: 3\sqrt{x}+6 : 2\sqrt{x}$, to find the values of x.

Ans. $x = 9$, or 4.

37. Given $x^2+11+\sqrt{x^2+11} = 42$, to find the values of x.

Ans. $x = \pm 5$, or $\pm\sqrt{38}$.

38. Given $(x-5)^3 - 3\,(x-5)^{\frac{3}{2}} = 40$, to find the values of x.

Ans. $x = 9$, or $(-5)^{\frac{2}{3}} + 5$.

39. Given $x + \sqrt{x+6} = 2 + 3\sqrt{x+6}$, to find the values of x.

ANS. $x = 10$, or -2.

40. Given $\overline{x^2+5}|^2 - 4x^2 = 160$, to find the values of x.

ANS. $x = \pm 3$, or $\pm\sqrt{-15}$.

41. Given $x^2 - 7x + \sqrt{x^2-7x+18} = 24$, to find the values of x.

ANS. $x = 9$, or -2, or $\dfrac{7 \pm \sqrt{173}}{2}$.

42. Given $9x - 4x^2 + \sqrt{4x^2-9x+11} = 5$, to find the values of x.

ANS. $x = 2$, or $\dfrac{1}{4}$, or $\dfrac{9 \pm \sqrt{-31}}{8}$.

43. Given $x^2 + \sqrt{5x+x^2} = 42 - 5x$, to find the values of x.

ANS. $x = 4$, or -9, or $\dfrac{-5 \pm \sqrt{221}}{2}$.

44. Given $\dfrac{2}{\overline{x+2}|^{\frac{3}{2}}} + \dfrac{\sqrt{x+2}}{2} = \dfrac{17}{4\sqrt{x+2}}$, to find the values of x.

ANS. $x = 6$, or $-\dfrac{3}{2}$.

45. Given $\dfrac{x}{x+4} + \dfrac{4}{\sqrt{x+4}} = \dfrac{21}{x}$, to find the values of x.

ANS. $x = 12$, or -3, or $\dfrac{49 \mp \sqrt{65}}{2}$.

46. Given $\frac{3x+5}{3x-5} + \frac{3x-5}{3x+5} = 2\frac{25}{176}$, to find the values of x.

Ans. $x = \pm 9$.

47. Given $x + \sqrt{x} + 2 = \frac{x^2 + x - 4}{\sqrt{x}}$, to find the values of x.

Ans. $x = 4$, or 1.

48. Given $\frac{x^2}{\overline{x^2-4}|^2} + \frac{6}{x^2-4} = \frac{351}{25x^2}$, to find the values of x.

Ans. $x = \pm 3$, or $\pm\sqrt{\frac{39}{11}}$.

49. Given $\overline{x+\frac{8}{x}}|^2 + x = 42 - \frac{8}{x}$, to find the values of x.

Ans. $x = 4$, or 2, or $\frac{-7 \pm \sqrt{17}}{2}$.

50. Given $x + 4 - 2\sqrt{\frac{x+4}{x-4}} = \frac{3}{x-4}$, to find the values of x.

Ans. $x = \pm 5$, or $\pm\sqrt{17}$.

51. Given $x^2 - \frac{5x}{2} + 15 = \frac{25x^2}{16} - \frac{64}{x^2}$, to find the values of x.

Ans. $x = 4$, or -8, or $\frac{-2 \pm \sqrt{-71}}{9}$.

52. Given $\frac{35\frac{5}{7}}{\sqrt{x^4 - 9x^2}} + \frac{\sqrt{x^2-9}}{7x} = \frac{19}{2x}$, to find the values of x.

Ans. $x = \pm 5$, or $\pm\frac{\sqrt{15661}}{2}$.

53. Given $3\,.\,\overline{\overline{x-1}|^2 - x}|^2 + 2x = 341 + 2\,.\,\overline{x-1}|^2$, to find the values of x.

Ans. $x = 5$, or -2; or $\dfrac{3\sqrt{3} \pm \sqrt{-109}}{2\sqrt{3}}$.

54. Given $x^4 + \dfrac{13x^3}{3} - 39x = 81$, to find the values of x.

Ans. $x = \pm 3$, or $\dfrac{-13 \pm \sqrt{-155}}{6}$.

55. Given $4x^4 + \dfrac{x}{2} = 4x^3 + 33$, to find the values of x.

Ans. $x = 2$, or $-\dfrac{3}{2}$; or $\dfrac{1 \pm \sqrt{-43}}{4}$.

56. Given $\overline{x-2}|^2 - 6x^{\frac{1}{2}}.(x-2) = 24 - 14x + 15x^{\frac{1}{2}}$, to find the values of x.

Ans. $x = 16$, or 1; or $\dfrac{\pm 3\sqrt{-11} - 1}{2}$.

57. Given $\overline{4x+1}|^2 + 4x^{\frac{1}{2}}.(4x+1) = 1912 - (10x + 3x^{\frac{1}{2}})$, to find the values of x.

Ans. $x = 9$, or $\dfrac{49}{4}$; or $\dfrac{-90 \mp \sqrt{-181}}{8}$.

58. Given $8x^2 - 13 = \dfrac{3x}{2} + \sqrt{6x^3 + 52x^2}$, to find the values of x.

Ans. $x = 2$, or $-\dfrac{13}{8}$; or $\dfrac{3 \pm \sqrt{13321}}{128}$.

59. Given $4x^2 + 21x + 8x^{\frac{1}{2}}\sqrt{7x^2 - 5x} = 207 - \frac{4x^2}{3}$, to find the values of x.

Ans. $x = 3$, or $\frac{207}{16}$, or $\frac{-129 \pm 3\sqrt{-2567}}{32}$.

60. Given $\frac{2x + \sqrt{x}}{2x - \sqrt{x}} = 2\frac{4}{15} - \frac{2x - \sqrt{x}}{2x + \sqrt{x}}$, to find the values of x.

Ans. $x = 4$.

V. *Adfected Quadratics involving two unknown Quantities.*

1. Given $x + 4y = 14$, and $y^2 + 4x = 2y + 11$, to find the values of x and y.

Ans. $\begin{cases} x = -46, \text{ or } 2, \\ y = 15, \text{ or } 3. \end{cases}$

2. Given $\frac{2x + 7y}{4x} = 2y - \frac{51 + 2x}{10}$, and $\frac{4x + 3y}{16} = y - 2$, to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } -\frac{56}{27}, \\ y = 4, \text{ or } \frac{640}{351}. \end{cases}$

3. Given $\frac{4xy + 2y - 1}{5x} - 1 = \frac{4y + 3x + 1}{5} - \frac{17 - x}{3}$,

and $\frac{3x + 2y}{7} = \frac{4x - 5y}{5} + 2$,

to find the values of x and y.

Ans. $\begin{cases} x = 5, \text{ or } -\frac{19}{210}, \\ y = 3, \text{ or } \frac{14453}{9450}. \end{cases}$

4. Given $x^2 + y : x^2 - y :: 9 : 7,$ and $1 + x^2 : y + 4 :: 5y + 7 : 3y,$ to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x = \pm 4, \text{ or } \pm 4\sqrt{-\frac{7}{19}}, \\ y = 2, \text{ or } -\frac{14}{19}. \end{cases}$$

5. Given $x^2 + 2x^3y = 441 - x^4y^2,$ and $xy = 3 + x,$ to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x = 3, \text{ or } -7, \text{ or } -2 \pm \sqrt{-17}, \\ y = 2, \text{ or } \frac{4}{7}, \text{ or } \frac{1 \pm \sqrt{-17}}{-2 \pm \sqrt{-17}}. \end{cases}$$

6. Given $x^2 + 4y^2 = 216 - 4xy,$ and $4y^2 - x^2 = 64,$ to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x = \pm 6, \\ y = \pm 5. \end{cases}$$

7. Given $\overline{x + y}|^2 - 3y = 28 + 3x,$ and $2xy + 3x = 36,$ to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x = 4, \text{ or } \frac{9}{2}, \text{ or } \frac{-11 \pm \sqrt{409}}{4}, \\ y = 3, \text{ or } \frac{5}{2}, \text{ or } \frac{-5 \mp \sqrt{409}}{4}. \end{cases}$$

8. Given $x^2 + 10x + y = 119 - 2\sqrt{y} \times (x + 5),$ and $x + 2y = 13,$ to find the values of x and y.

$$\text{Ans.}\ \begin{cases} x = 5, \text{ or } \frac{17}{2}, \text{ or } \frac{-69 \mp \sqrt{241}}{4}, \\ y = 4, \text{ or } \frac{9}{4}, \text{ or } \frac{121 \pm \sqrt{241}}{8}. \end{cases}$$

9\. Given $\dfrac{x^4}{y^2} + \dfrac{2x^2}{y} = 4\frac{8}{49}$, and $x^2 + y^2 = 65$, to find the values of x and y.

ANSWER,

$$x = \pm 4, \text{ or } \pm \frac{4\sqrt{-65}}{7}; \text{ or } \pm \frac{\sqrt{-450 \mp 30\sqrt{3410}}}{7},$$

$$y = 7, \text{ or } -\frac{65}{7}; \text{ or } \frac{15 \pm \sqrt{3410}}{7}.$$

10\. Given $x + y + \sqrt{x+y} = 6$, and $x^2 + y^2 = 10$, to find the values of x and y.

ANS. $\begin{cases} x = 3, \text{ or } 1; \text{ or } \dfrac{9 \pm \sqrt{-61}}{2}, \\ y = 1, \text{ or } 3; \text{ or } \dfrac{9 \mp \sqrt{-61}}{2}. \end{cases}$

11\. Given $x^2 + 5\sqrt{x^2 + 2y + 3} = 47 - 2y$,
and $6x - 5y = 9$,
to find the values of x and y.

ANS. $\begin{cases} x = 4, \text{ or } -\dfrac{32}{5}; \text{ or } \dfrac{-6 \pm \sqrt{2551}}{5}, \\ y = 3, \text{ or } -\dfrac{237}{25}; \text{ or } \dfrac{-81 \pm \sqrt{2551}}{25}. \end{cases}$

12\. Given $x^2 + 3x + y = 73 - 2xy$, and $y^2 + 3y + x = 44$, to find the values of x and y.

ANS. $\begin{cases} x = 4, \text{ or } 16; \text{ or } -14 \pm \sqrt{58}, \\ y = 5, \text{ or } -7; \text{ or } -1 \pm \sqrt{58}. \end{cases}$

13\. Given $\dfrac{y}{(x+y)^{\frac{3}{2}}} + \dfrac{\sqrt{x+y}}{y} = \dfrac{17}{4\sqrt{x+y}}$, and $x = y^2 + 2$, to find the values of x and y.

ANS. $\begin{cases} x = 6, \text{ or } 3, \text{ or } \dfrac{9 \mp 3\sqrt{-119}}{32}, \\ y = 2, \text{ or } 1, \text{ or } \dfrac{-3 \pm \sqrt{-119}}{8}. \end{cases}$

14. Given $\left.\begin{array}{l} y - y^{\frac{1}{2}} = 16 - x, \\ \text{and } 28 - y = x + 4x^{\frac{1}{2}}, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 4, \text{ or } \left[\dfrac{58}{17}\right]^2, \\ y = 16, \text{ or } \dfrac{784}{289}. \end{cases}$$

15. Given $\left.\begin{array}{l} x^4 + y^4 = 97, \\ \text{and } x + y = 5, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 3, \text{ or } 2; \text{ or } \dfrac{5 \pm \sqrt{-151}}{2}, \\ y = 2, \text{ or } 3; \text{ or } \dfrac{5 \mp \sqrt{-151}}{2}. \end{cases}$$

16. Given $\left.\begin{array}{l} \sqrt{\dfrac{3x - 2y}{2x}} + \sqrt{\dfrac{2x}{3x - 2y}} = 2, \\ \text{and } x^2 - 18 = x \,.\, (4y - 9), \end{array}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 6, \text{ or } 3, \\ y = 3, \text{ or } \dfrac{3}{2}. \end{cases}$$

17. Given $x + 4\sqrt{x} + 4y = 21 + 8\sqrt{y} + 4\sqrt{xy}$,
and $\sqrt{x} + \sqrt{y} = 6$,
to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 25, \text{ or } \dfrac{25}{9}, \\ y = 1, \text{ or } \dfrac{169}{9}. \end{cases}$$

18. Given $\left.\begin{array}{l} x + y = 5, \\ \text{and } (x^2 + y^2) \times (x^3 + y^3) = 455, \end{array}\right\}$ to find the values of x and y.

$$\text{Ans. } \begin{cases} x = 3, \text{ or } 2; \text{ or } \dfrac{5}{2} \pm \dfrac{1}{2}\sqrt{-\dfrac{103}{3}}, \\ y = 2, \text{ or } 3; \text{ or } \dfrac{5}{2} \mp \dfrac{1}{2}\sqrt{-\dfrac{103}{3}}. \end{cases}$$

19. Given $x+y+\sqrt{\dfrac{x+y}{x-y}}=\dfrac{6}{x-y}$ and $x^2+y^2=41$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=\pm 5, \text{ or } \pm 3\sqrt{\dfrac{5}{2}}, \\ y=\pm 4, \text{ or } \pm\sqrt{\dfrac{37}{2}}. \end{cases}$$

20. Given $\dfrac{x^4}{y^2}+\dfrac{y^4}{x^2}=136\dfrac{1}{9}-2xy$, and $x+4=14-y$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=6, \text{ or } 4; \text{ or } 5\pm 5\sqrt{-\dfrac{13}{11}}. \\ y=4, \text{ or } 6; \text{ or } 5\mp 5\sqrt{-\dfrac{13}{11}}. \end{cases}$$

21. Given $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}=4\dfrac{4}{5}$, and $\sqrt{\dfrac{x-y}{x^4}}+\dfrac{1}{x}=\dfrac{4}{9\sqrt{x-y}}$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=3, \text{ or } \dfrac{45}{2}; \text{ or } \dfrac{3}{16}, \text{ or } \dfrac{45}{32}, \\ y=2, \text{ or} -\dfrac{135}{4}; \text{ or } \dfrac{1}{8}, \text{ or} -\dfrac{135}{64}. \end{cases}$$

22. Given $\sqrt{6\sqrt{x}+6\sqrt{y}}+\dfrac{1}{2}\sqrt{x}=9-\dfrac{1}{2}\sqrt{y}$,
and $x-y=12$.
to find the values of x and y.

$$\text{Ans.}\begin{cases} x=16, \text{ or } \dfrac{784}{81}, \\ y=4, \text{ or } \dfrac{676}{81}. \end{cases}$$

23. Given $y^4-432=12xy^2$, and $y^2=12+2xy$, to find the values of x and y.

$$\text{Ans.}\begin{cases} x=2, \\ y=6. \end{cases}$$

24. Given $\frac{4}{y^2} + \frac{4+y}{y} = \frac{8+4y}{x} + \frac{12y^2}{x^2}$; and $4y^2 - xy = x$, to find the values of x and y

$$\text{Ans.} \begin{cases} x = 2, \text{ or } -\frac{50}{3}, \\ y = 1, \text{ or } -\frac{5}{3}. \end{cases}$$

25. Given $\sqrt{\overline{1+x}|^2 + y^2} + \sqrt{\overline{1-x}|^2 + y^2} = 4$,
and $\overline{4-x^2}|^2 = 18 - 4y^2$,
to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 1, \text{ or } \pm\sqrt{10}, \\ y = \pm \frac{3}{2}, \text{ or } \pm 3\sqrt{-\frac{1}{2}}. \end{cases}$$

26. Given $\frac{x+\sqrt{x+y}}{x-\sqrt{x+y}} - \frac{\sqrt{x}-x-y}{\sqrt{x}+x+y} = 2\frac{9}{40}$,
and $y^2 - \sqrt{xy^2} = \frac{4x}{9}$,
to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 9, \text{ or } \frac{196}{9}, \text{ or } \frac{289}{9}, \text{ or } 16, \\ y = 4, \text{ or } \frac{-14}{9}, \text{ or } -\frac{68}{9}, \text{ or } -\frac{4}{3}. \end{cases}$$

27. Given $\frac{x+\sqrt{x^2-y^2}}{x-\sqrt{x^2-y^2}} = 4\frac{1}{4} - \frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$,
and $x.(x+y) = 52 - \sqrt{x^2+xy+4}$,
to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 5, \text{ or } \pm \frac{10}{\sqrt{3}}, \\ y = \pm 4, \text{ or } \pm \frac{8}{\sqrt{3}}. \end{cases}$$

28. Given $\frac{y}{x} \cdot \sqrt{\frac{x}{y}} + \frac{1}{2} \cdot \sqrt{\frac{x}{y}} \cdot \sqrt[4]{\frac{y^3}{x^3}} = 5,$

and $\frac{2x^2}{y} - \frac{x}{3\sqrt{y}} = \frac{1}{3},$

to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 4, \text{ or } \frac{16}{9}, \text{ or } \frac{625}{64}, \text{ or } \frac{625}{144}, \\ y = 64, \text{ or } \frac{256}{9}, \text{ or } \left|\frac{625}{32}\right|^2, \text{ or } \left|\frac{-625}{48}\right|^2. \end{cases}$$

29. Given $x^2 - y^2 = 3,$

and $\overline{x^4 + y^4}|^2 + x^2y^2 . \overline{x^2 - y^2}|^2 + x^2 - y^2 = 328,$

to find the values of x and y.

$$\text{Ans.} \begin{cases} x = \pm 2, \text{ or } \pm 2\sqrt{-1}, \text{ or } \pm \sqrt{\frac{3 \pm 2\sqrt{-13}}{2}}, \\ y = \pm 1, \text{ or } \pm 2\sqrt{-1}, \text{ or } \pm \sqrt{\frac{-3 \pm 2\sqrt{-13}}{2}}. \end{cases}$$

30. Given $\frac{2y^2 - 8\sqrt{x}}{\sqrt{x}} + \sqrt{4y^2 - 16\sqrt{x}} = \frac{3\sqrt{x}}{2},$

and $\sqrt{x} + \sqrt{8 . (y - \sqrt{x}) - 4} = y + 1,$

to find the values of x and y.

$$\text{Ans.} \begin{cases} x = 4, \text{ or } \frac{4}{9}, \text{ or } \frac{4}{25}, \text{ or } -\frac{4}{3} \mp 16\sqrt{-\frac{13}{3}}, \text{ or } \frac{788 \pm 2\sqrt{644}}{625}, \\ y = 3, \text{ or } \frac{5}{3}, \text{ or } -1, \text{ or } \frac{7}{5}, \text{ or } 1 \pm 2\sqrt{-\frac{13}{3}}, \text{ or } \frac{137 \pm \sqrt{644}}{25}. \end{cases}$$

VI. *Problems producing Simple Equations, involving only one unknown Quantity.*

1. What number is that, from the treble of which if 18 be subtracted, the remainder is 6. *Ans.* 8.

2. What number is that, the double of which exceeds four-fifths of its half by 40. *Ans.* 25.

3. In fencing the side of a field, whose length was 450 yards, two workmen were employed; one of whom fenced 9 yards, and the other 6 *per* day. How many days did they work?

Ans. 30.

4. A mercer bought 4 pieces of silk, which together measured 50 yards; the second was twice, the third three times, and the fourth four times as long as the first. What were the respective lengths of the pieces?

Ans. 5, 10, 15, 20 yards.

5. A farmer sold 13 bushels of barley at a certain price; and afterwards 17 bushels at the same rate; and at the second time received 36 shillings more than at the first. What was the price of a bushel?

Ans. 9 shillings.

6. A person bought 198 gallons of beer, which exactly filled 4 casks; the first held twice as much as the second, the second twice as much as the third, and the third three times as much as the fourth. How many gallons did each hold?

Ans. 108, 54, 27, and 9 gallons.

7. A silversmith has 3 pieces of metal, which together weigh 48 ounces. The second weighs 12 ounces more than the first, and the third 9 ounces more than the second. What are their respective weights?

Ans. 5, 17, and 26 ounces.

8. A vintner fills a cask, containing 96 gallons, with a mixture of brandy, wine, and water. There are 20 gallons of water more than of brandy, and 17 more of wine than of water. How many are there of each?

Ans. 13 gallons of brandy, 33 of water, and 50 of wine.

of the brandy by 9 gallons, at 15 shillings a gallon, I find that I paid 1 shilling more for the brandy than for the rum. How many gallons were there of each?

Ans. 34 of brandy, and 43 of rum.

21. Two persons, *A* and *B*, have each an annual income of £.400. *A* spends every year £.40 more than *B*, and at the end of 4 years, the amount of their savings is equal to one year's income of either. What does each spend annually?

Ans. £.370, and £.330, respectively.

22. A draper sold two pieces of cloth, by one of which he lost £.6 more than by the other; and his whole loss was £.5 less than treble the less loss. What were the losses sustained by each piece?

Ans. £.11, and £.17.

23. A person engaged to reap a field of corn for 5 shillings an acre, but leaving 6 acres not reaped, he received £2. 10*s*. Of how many acres did the field consist?

Ans. 16.

24. In a naval engagement, the number of ships taken was 7 more, and the number burnt 2 fewer, than the number sunk. Fifteen escaped, and the fleet consisted of 8 times the number sunk. Of how many did the fleet consist?

Ans. 32.

25. A farmer hires a farm for £.175. 15*s*. a year; part of which he is not allowed to plough: he gives £.2 *per* acre for the arable, and for the rest, which was 5 acres less, he gives £1. 5*s*. *per* acre. How many acres were arable, and how many not?

Ans. 56 acres arable, 51 not?

26. A cistern is filled in twenty minutes by three pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third, *per* minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

Ans. 22, 7, and 12 gallons.

27. *A* and *B* began to play; *A* with exactly $\frac{4}{9}$ of the sum which *B* had. After winning £.10, he found that they had each the same sum. What had each at first?

Ans. A had £.16, and *B* £.36.

28. A person has a certain number of horses at livery stables, and three times as many at grass. He keeps 15 in constant employment; and his whole number is seven times the number in the stables. Required the whole number.

Ans. 35.

29. Two men at the distance of 150 miles set out to meet each other, one goes 3 miles in the time the other goes 7. What part of the distance does each travel?

Ans. One 45, and the other 105 miles.

30. *A* and *B* begin trade, *A* with triple the stock of *B*. They each gain £.50, which makes their stocks in the proportion of 7 to 3. What were their original stocks?

Ans. A's was £.300, and *B*'s £.100.

31. A charitable person distributed £.5. 14*s*. amongst some poor women and children, giving to each woman 6 shillings, and to each child two; and the number of women was to the number of children as 4 : 7. How many were relieved?

Ans. 12 women, and 21 children.

32. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$. Required the numbers.

Ans. 30 and 40.

33. A farmer has a stack of hay, from which he sells a quantity which is to the quantity remaining in the proportion of 4 to 5. He then uses 15 loads, and finds that he has a quantity left which is to the quantity sold as 1 to 2. How many loads did the stack at first contain?

Ans. 45.

34. There are 3 pieces of cloth, whose lengths are in the proportion of 3, 5, and 7; and 6 yards being cut off from each, the whole quantity is diminished in the proportion of 20 to 17. Required the length of each piece at first.

Ans. 24, 40, and 56 yards.

35. The number of days that 4 workmen were employed were severally as the numbers 4, 5, 6, 7; their daily wages were the same, viz. 3 shillings, and the sum received by the first and second was 36 shillings less than that received by the third and fourth. How much did each receive?

Ans. 36, 45, 54, and 63 shillings.

36. From two casks of equal size are drawn quantities, which are in the proportion of 6 to 7; and it appears that if 16 gallons less had been drawn from that which is now the emptier, only half as much would have been drawn from it as from the other. How many gallons were drawn from each?

Ans. 24 and 28.

37. On the inclosure of a parish, a proprietor had for his allotment two pieces of land, which were of the form of rectangular parallelograms. The longer sides of the parallelograms were in the ratio of 6 to 11, and the adjacent sides of the less as 3 to 2. The periphery of the less was 135 yards more than the longer side of the greater. Required the sides of the less, and the longer side of the greater.

Ans. The sides of the less were 90 and 60; and the longer side of the greater was 165 yards.

38. Two persons *A* and *B* travelling, each with £.80, meet with robbers, who take from *A* twice as much as from *B*, and £.5 over, and leave *A* within £.13 half as much as *B*. How much is taken from each?

Ans. 69, and 32 pounds.

39. A person distributes forty shillings amongst 50 people; giving to some nine-pence each, and to the rest fifteen pence. How many were there of each?

Ans. 45, and 5.

40. A person put out a certain sum to interest for $6\frac{1}{4}$ years, at 5 *per* cent. simple interest, and found that if he had put out the same sum for 12 years and 9 months at 4 *per* cent. he would have received £.185 more. What was the sum put out?

Ans. £.1000.

41. Two persons, *A* and *B*, were partners. *A*'s money remained in the firm 6 years, and his gain was one-fourth of his principal; and *B*'s money, which was £.50 less than *A*'s, had been in the firm 9 years, when they dissolved partnership, and it appeared that if *B* had gained £.6. 5*s.* less, his gain and principal would have been to *A*'s gain

and principal as 4 to 5. What was the principal of each?

Ans. £.200, and £.150.

42. A merchant bought wheat at the rate of £.3. 10*s*. for 5 bushels. He afterwards bought some inferior, which was in quantity to the former as 3 to 4, at the rate of £.3. 12*s*. for 8 bushels; and sold the whole for 10 shillings a bushel; in consequence of which, he lost £.7. 16*s*. by the bargain. How much of each did he buy?

Ans. 48 bushels of the better, and 36 of the worse.

43. Three persons, *A*, *B*, and *C*, spent equal sums at a tavern. *C* having no money, the reckoning was paid by *A* and *B*. When *C* came to reimburse them, he paid 4 times as much to *A* as to *B*, and observed, that if *B* had paid 3 shillings more of his reckoning, their demands would have been equal. Required the sum each spent, and the respective parts of *C*'s reckoning that *A* and *B* paid.

Ans. Each spent 10 shillings; *A* paid 8, and *B* 2.

44. A farmer had 2 flocks of sheep, one of which contained 40, and the other was sold for £.30; but one sheep of the latter was worth four of the other; and the value of the first flock was only £.4 more than the price of eight sheep of the second. How many sheep did the second flock contain, and what was the value of a sheep of each?

Ans. The number was 15, and the prices £.2, and 10 shillings.

45. *A* and *B* playing at billiards, *A* bet 5 shillings to 4 on every game, and found that after a certain number of games he had won 10 shillings. Had *B* won one game more, the number won by him would have been

to the number won by A as 3 to 4. How many did each win?

Ans. A won 20, and B 14.

46. A besieged garrison had such a quantity of bread, as would, if distributed to each at 10 ounces a day, last 6 weeks; but having lost 1200 men in a sally, the governor was enabled to encrease the allowance to 12 ounces *per* day for 8 weeks. Required the number of men at first in the garrison.

Ans. 3200.

47. A composition of copper and tin containing 100 cubic inches weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces.

Ans. 420 of copper, and 85 of tin.

48. There are two towns, A and B, which are 131 miles distant from each other. A coach sets out from A at 6 o'clock in the morning, and travels at the rate of 4 miles an hour without intermission, in the direct road towards B. At 2 o'clock in the afternoon of the same day a coach sets out from B to go to A, and goes at the rate of 5 miles an hour constantly. Where will they meet?

Ans. 76 miles from A, and 55 from B.

49. Out of a certain sum, a man paid his creditors £.96; half of the remainder he lent his friend; he then spent one-fifth of what now remained; and after all these deductions had one-tenth of his money left. How much had he at first?

Ans. £.128.

50. A and B, in order to keep up the price of copper

to £.86 *per* ton, agree for a certain time to sell all the copper they raise, jointly; yet so that each shall be paid proportionably to the quantity he raises. Now the whole quantity raised in the stipulated time was 235 tons; and *A* receives £.4214 more than *B*. Required the quantity raised by each.

Ans. 142 tons by *A*, and 93 by *B*.

51. Bought two pieces of linen, one of which wanted 12 yards of being four times as long as the other. The longer cost 5 shillings, and the shorter 4 shillings a yard; 23 yards being cut off from the longer, and 5 from the shorter, and the remainders being sold for 1 shilling a yard more than they cost, I received £.7. 2*s*. How many yards of each were there?

Ans. 40, and 13.

52. A courier, passing through a certain place *A*, travels at the rate of 13 miles in 2 hours; 12 hours afterwards another passes through the same place, travelling the same road, at the rate of 26 miles in 3 hours. How long, and how far must he travel before he overtakes the first.

Ans. 36 hours, and 312 miles.

53. As *A* and *B* were going to school, *A* first shot an arrow in the direction in which they were going, which *B* took up and shot forward; and so on alternately till the arrow had passed exactly from one mile-stone to another; when it appeared that *A* had shot the arrow 8 times, and *B* 7 times. Some time afterwards, *A* and *B* were on the opposite banks of a river, the breadth of which they wished to ascertain; *A* first shot the arrow across the river, and it flew 13 yards beyond the bank on which *B* stood; *B* then took it up, and from the place where it had fallen, shot it back across the river: it now

fell $9\frac{6}{7}$ yards beyond the bank upon which *A* stood. Required the breadth of the river.

Ans. 100 yards.

54. Three merchants, *A*, *B*, and *C*, enter into a speculation; *B* subscribes £.10 more than four-fifths of what *A* does; and *C* £.30 more than half of what *B* does. *A*'s gain is two-fifths of his subscription, and *B*'s is £.148. What are the respective sums subscribed, and the whole gain?

Ans. The sums subscribed are 450, 370, and 215 pounds; and the whole gain is £.414.

55. There are two places, 154 miles distant, from which two persons set out at the same time to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in four hours. How long, and how far did each travel before they met.

Ans. 56 hours; and 84, and 70 miles.

56. *A* sets out from a certain place, and travels at the rate of 7 miles in five hours; and eight hours afterwards *B* sets out from the same place, and travels the same road at the rate of 5 miles in three hours. How long, and how far must *A* travel before he is overtaken by *B*.

Ans. 50 hours, and 70 miles.

57. A man lent out a certain sum to interest at £.8 *per* cent. *per* annum. He suffered this to accumulate at simple interest for 12 years; and then putting out the principal and interest at the same rate found that the present annual interest exceeded the former by £.38. 8*s*. Required the sum put out each time.

Ans. £.500 the first time, and £.980 the second.

58. A waterman finds by experience that he can with the advantage of a common tide row down a river from *A* to *B* which is 18 miles in an hour and a half, and that to return from *B* to *A* against an equal tide, though he rows back along the shore, where the stream is only three-fifths as strong as in the middle, takes him just two hours and a quarter. It is required from hence to find at what rate *per* hour the tide runs in the middle where it is strongest.

Ans. At the rate of two miles and a half *per* hour.

59. The ingredients of a loaf of bread are rice, flour, and water, and the weight of the whole is 15lbs. The weight of the rice augmented by 5lbs. is two-thirds of the weight of the flour, and the weight of the water is one-fifth of the weight of the flour and rice together. Required the weight of each.

Ans. Rice 2lbs., flour $10\frac{1}{2}$lbs., water $2\frac{1}{2}$lbs.

60. Suppose two fingers of a watch (*a*) and (*b*) were together on Sunday noon at 12 o'clock, and that the motion of each was such that (*a*) moved round the horarycircle in one hour, and *b* in $1\frac{1}{60}$ hour. When will they be together again for the first time?

Ans. 61 hours.

61. A draper bought a piece of cloth for £.69, from which he cut off 11 yards. He then met with another piece of equal goodness for which he gave £.21, and found that if it had been one yard longer, its length would have been to the length of the remainder of the first as 2 to 3. How many yards were there in each piece, and what was the price of a yard?

Ans. 23 in the first, and 7 yards in the second; and the price £.3 *per* yard.

62. Divide the number 198 into 5 such parts, that the

first increased by one, the second increased by two, the third diminished by three, the fourth multiplied by four, and the fifth divided by five, may be all equal.

Ans. 23, 22, 27, 6, and 120, are the numbers.

63. *A* and *B* set out from two places, *C* and *D*, at the same time, towards *E*; the road from *C* to *E* being through *D*. *A* travels 7 miles an hour, and at that rate of travelling would have overtaken *B* 5 miles before he got to *E*; but after arriving at *D*, he travels $6\frac{3}{5}$ miles an hour, in consequence of which he overtakes *B* just as he enters *E*. Supposing *B* to travel 5 miles an hour, what are the distances between *C*, *D*, and *E*.

Ans. From *C* to *D* 14 miles, and from *D* to *E* 40.

64. A gentleman wishing his two daughters to receive equal portions when they became of age, bequeathed to the elder the accumulated insterest of a certain sum of money, bought at the time of his death into the 4 *per* cent. stock at 88; and to the younger the accumulated interest of a sum less than the former by £.3500, bought at the same time into the 3 *per* cents. at 63. Supposing their ages at the time of their father's death to have been 17 and 14, what would be the sum bought into the stocks in each case, and what would be the fortune of each?

Ans. The sums would be £.7700, and £.4200; and fortune £.1400.

65. Out of a common pack of cards, a certain number, including the ten of diamonds, was dealt equally amongst four persons, the dealer turning up the last card, which was the ten of spades, which he gives himself. Now if twice the number of cards had been dealt to each, the ten of spades being turned up by the dealer, and the ten of diamonds being still dealt out, the chance of the dealer's

having the ten of diamonds would be to the chance against him as 3 : 10. Required the number of cards dealt to each the second time.

Ans. 10.

66. Two companies of soldiers consisting of equal numbers were sent out under *A* and *B*, from two hostile camps, to reconnoitre. Falling in with each other, a skirmish ensued, in which *A* lost 50 killed and prisoners, and *B* had 20 killed. *A* however having being reinforced by a party equal to five-sevenths of the number which *B* had remaining; and *B* having been reinforced by a number greater by 46 than three-fifths of the number which *A* had remaining, they renewed the engagement, when *A* was forced to retire with the additional loss of 30 men. When the returns were made, *B* found he had again lost 20 men, but that he had then twice as many men remaining as *A* had. How many had each at first?

Ans. 90.

67. A sportsman, who kept an account of the number of birds which he killed, found that each succeeding season he wanted 50, in order that the number killed might bear the proportion of 3 : 2 to the number killed in the preceding year. In the fourth year he found that he had killed 170 fewer than three times the number killed in the first year. How many did he kill the first year?

Ans. 180.

68. Four men walking abroad found a purse containing shillings only, out of which every one of them took a number at a venture. Afterwards comparing their numbers together, they found that if the first took 25 shillings from the second, it would make his number equal to what the second had left. If the second took 30

shillings from the third, his money would then be triple what the third had left. And if the third took 40 shillings from the fourth, his money would then be double of what the fourth had left. Lastly, the fourth taking 50 shillings from the first, he would then have three times as much as the first had left, and 5 shillings over. What had each?

Ans. 100, 150, 90, and 105 shillings, respectively.

69. Fifteen current guineas should weigh 4 ounces; but a parcel of light gold being weighed and counted, was found to contain 9 more guineas, than was supposed from the weight; and a part of the whole, exceeding the half by 10 guineas and a half, was found to be $1\frac{1}{4}$ oz. deficient in weight. What was the number of guineas?

Ans. 189.

70. A merchant bought a quantity of wheat for £.200, half of which he reserved for his private use. He then sold 5 bushels more than $\frac{3}{4}$ of the remaining quantity at such a price as to gain £.40 *per* cent. But the price of wheat having advanced, he sold the remainder at such a price as to gain £.67 *per* cent. by what he then sold. And had the whole been sold at this latter price he would have gained £.160 *per* cent. How much did he buy, and how did he sell it?

Ans. He bought 400 bushels; and sold the first portion at 14*s.*, and the second at 26*s.* *per* bushel.

VII. *Problems producing Simple Equations, involving two unknown Quantities.*

1. A draper bought two pieces of cloth for £.12. 13*s.*; one being 8*s.*, and the other 9*s.* *per* yard. He sold them

each at an advanced price of 2*s.* *per* yard, and gained by the whole £.3. What were the lengths of the pieces?

Ans. 17 yards of the first, and 13 of the second.

2. A bill of £.26. 5*s.* was paid with half guineas and crowns, and twice the number of half guineas exceeded three times the number of crowns by 17. How many were there of each?

Ans. 40 half guineas, and 21 crowns.

3. Two labourers, *A* and *B*, received £.5. 17*s.* for their wages; *A* having been employed 15, and and *B* 14 days; and *A* received for working four days 11*s.* more than *B* did for three days. What were their daily wages?

Ans. *A* had 5*s.*, and *B* 3*s.* a day.

4. A person had two casks, the larger of which he filled with ale, and the smaller with cyder. Ale being half a crown, and cyder 1*s.* *per* gallon, he paid £.8. 16*s.*; but had he filled the larger with cyder, and the smaller with ale, he would have paid £.11. 5*s.* 6*d.* How many gallons did each hold?

Ans. The larger contained 18, and the smaller 11 gallons.

5. A person expends half a crown in apples and pears, buying his apples at 4, and his pears at 5 a penny; and afterwards accommodates his neighbour with half his apples and one-third of his pears for 13 pence. How many did he buy of each?

Ans. 72 apples, and 60 pears.

6. Two persons, *A* and *B*, played cards, each with a different sum. After a certain number of games, *A* had won half as much as he had at first, and found that if he had 15*s.* more he would have had just three times as

much as *B*. But *B* afterwards won 10*s*. back, and he had then twice as much as *A*. What had each at first?

Ans. *A* had 14, and *B* 19 shillings.

7. A certain sum of money put out to interest, amounts in 8 months to £.297. 12*s*.; and in 15 months its amount is £.306 at simple interest. What is the sum, and the rate *per* cent.?

Ans. £.288, at 5 *per* cent.

8. A farmer being asked how many quarters of wheat he had sold in the market, answered if he had sold 8 quarters more, and got 7*s*. *per* quarter more than he did, he should have received £.11. 15*s*. more than he had: but if he had sold 7 quarters more at 8*s*. *per* quarter more, he should have had £.11. 17*s*. more. How many quarters did he sell, and what was the price?

Ans. 13 quarters, at 11*s*. *per* quarter.

9. There is a number consisting of two digits, the second of which is greater than the first; and if the number be divided by the sum of its digits, the quotient is 4: but if the digits be inverted, and that number divided by a number greater by 2 than the difference of the digits, the quotient becomes 14. Required the number.

Ans. 48.

10. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$?

Ans. $\frac{4}{5}$.

11. A farmer parting with his stock, sells to one person 9 horses and 7 cows for £.300; and to another, at the same prices, 6 horses and 13 cows for the same sum. What was the price of each?

Ans. The price of a cow was £.12, and of a horse £.24.

12. A farmer hires a farm for £.245 *per ann.*, the arable land being valued at £.2 an acre, and the pasture at 28 shillings: now the number of acres of arable is to half the excess of the arable above the pasture as 28 : 9. How many acres were there of each?

Ans. 98 acres of arable, and 35 of pasture.

13. A person owes a certain sum to two creditors. At one time he pays them £.53, giving to one $\frac{4}{11}$ of the sum which is due, and to the other £.3 more than $\frac{1}{6}$ of his debt to him. At a second time he pays them £.41, giving to the first $\frac{3}{7}$ of what was due to him, and to the other $\frac{1}{3}$ of what was due to him. What were the debts?

Ans. £.121, and £.36.

14. *A* and *B* playing at backgammon, *A* bet 3*s*. to 2*s*. on every game, and after a certain number of games found that he had lost 17 shillings. Now had *A* won 3 more from *B*, the number he would then have won, would have been to the number *B* would have won as 5 to 4. How many games did they play?

Ans. 9.

15. A vintner has two casks of wine, from the greater of which he draws 15 gallons, and from the less 11; and finds the quantities remaining in the proportion of 8 to 3. After they become half empty, he puts 10 gallons of water into each, and finds that the quantities of liquor now in them are as 9 to 5. How many gallons will each hold?

Ans. The larger 79, and the smaller 35 gallons.

16. A person having laid out a rectangular bowling-green, observed that if each side had been 4 yards longer, the adjacent sides would have been in the ratio of 5 to 4; but if each had been 4 yards shorter, the ratio would have been 4 to 3. What are the lengths of the sides?

Ans. 36, and 28 yards.

17. At an election for two members of parliament, three men offer themselves as candidates, and all the electors give single votes. The numbers of voters for the two successful ones are in the ratio of 9 to 8; and if the first had had 7 more, his majority over the second would have been to the majority of the second over the third as 12 : 7. Now if the first and third had formed a coalition, and had one more voter, they would each have succeeded by a majority of 8. How many voted for each?

Ans. 369, 328, and 300, respectively.

18. Two shepherds, *A* and *B*, are intrusted with the charge of two flocks of sheep. *A*'s consisting chiefly of ewes, many of which produced lambs, is at the end of the year increased by 80; but *B* finds his flock diminished by 20; when their numbers are in the proportion of 8 to 3. Now had *A* lost 20 of his sheep, and *B* had an increase of 90, the numbers would have been in the proportion of 7 to 10. What were the numbers?

Ans. *A*'s 160, and *B*'s 110.

19. Two persons, *A* and *B*, can perform a piece of work in 16 days. They work together for 4 days, when *A* being called off, *B* is left to finish it, which he does in 36 days more. In what time would each do it separately?

Ans. *A* in 24 days, and *B* in 48 days.

20. There is a cistern, into which water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, $\frac{5}{12}$ of the cistern is filled in four hours; and if one of the equal cocks be stopped, $\frac{7}{9}$ of the cistern is filled in ten hours and forty minutes. In how many hours would each cock fill the cistern?

Ans. Each of the equal ones in 32 hours, and the other in 28.

21. Some hours after a courier had been sent from *A* to *B*, which are 147 miles distant, a second was sent, who wished to overtake him just as he entered *B*; in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles added to the time in which the second travels 56 miles is 13 hours and 40 minutes. How many miles does each go *per* hour?

Ans. The first goes 3, and the second 7 miles an hour.

22. Two loaded waggons were weighed, and their weights were found to be in the ratio of 4 to 5. Parts of their loads, which were in the proportion of 6 to 7 being taken out, their weights were then found to be in the ratio of 2 to 3; and the sum of their weights was then ten tons. What were the weights at first?

Ans. 16, and 20 tons.

23. A gentleman gave away a certain sum in charity to 14 men and 15 women. Had the sum been less by 12 shillings, and only half the number of men relieved, the rest being divided amongst the women, each woman would have received two shillings more than each man did. But if there had been only 8 women, and the rest had been divided amongst the men, each man would have received twice as much as each woman. How much money was given away?

Ans. 24 guineas.

24. When wheat was 5 shillings a bushel, and rye 3 shillings, a man wanted to fill his sack with a mixture of rye and wheat for the money he had in his purse. If he bought 7 bushels of rye, and laid out the rest of his money in wheat, he would want 2 bushels to fill his sack; but if he bought 6 bushels of wheat, and filled his sack with rye, he would have 6 shillings left. How must he lay out his money, and fill his sack?

Ans. He must buy 18 bushels of wheat, and 12 bushels of rye.

25. A draper bought two pieces of cloth of different kinds for £.37. 4*s.*: there were 6 yards of the coarser more than there were of the finer; and had the coarser cost 2 shillings a yard more than it did, 6 yards of the coarser would have cost just as much as 5 yards of the finer. He afterwards bought 4 yards of the finer, and 12 of the coarser at the same prices *per* yard, and found their value less than that of the former pieces in the ratio of 20 : 31. How many yards did he buy the first time, and what did he give *per* yard for each?

Ans. 9 yards of the finer, and 15 of the coarser; and the prices were 36, and 28 shillings *per* yard.

26. A mercer bought two pieces of silk of different lengths for £.50.; the price of two yards of the shorter was 6*s*. 8*d*. more than the price of 3 yards of the longer; and each piece cost the same sum. He cut off two yards from each, and sold the rest for £.53. 12*s*. Now if he had sold the whole at that rate, he would have gained £.5 by each piece. How many yards did each piece contain?

Ans. 25, and 15 yards.

27. *A* sets out express from *C* towards *D*, which is 112 miles distant, and travels a certain number of miles *per* hour. Three hours afterwards *B* sets out from *D* towards *C*, and goes 2 miles an hour more than *A*, and when they meet, finds that he has travelled 60 miles. Now had *A* travelled five hours less, and *B* gone 2 miles an hour more, he would have travelled 80 miles at their meeting. How many miles did each go *per* hour, and how many hours did they travel before they met?

Ans. *A* went 4, and *B* 6 miles an hour, and they travelled 10 hours after *B* set out.

28. *A* and *B* engaged to reap a field of corn in 12 days. The times in which they could severally reap an acre are as 2 : 3. After some time, finding themselves unable to finish it in the stipulated time, they called in *C* to help them; whose rate of working was such, that had he wrought with them from the beginning, it would have been finished in 9 days. Also the times in which he could severally have reaped the field with *A* alone, and with *B* alone, are in the proportion of 7 to 8. When was *C* called in?

Ans. After 6 days.

29. Two mixtures are made of brandy and sherry; the quantities of brandy in each being as 4 to 3; and the difference of the quantities of sherry being greater by

25 gallons than the difference of the quantities of brandy. Also, if three times the quantity of brandy had been put into the first mixture, and twice the quantity into the second, the quantities of brandy would have been proportional to the quantities of sherry. But if the sherry in the second mixture had been mixed with the brandy in the first, and the sherry in the first with the brandy in the second, the whole mixtures would then have been in the ratio of 5 to 6. Required the quantities of brandy and sherry in each mixture.

Ans. The quantities of brandy are 80, and 60 gallons, and the quantities of sherry are 90, and 45 gallons.

30. Two Spanish muleteers, *A* and *B*, were seated under a tree in order to dine; and on examining, found their stock of provisions to consist of 5 small loaves of bread, three of which were *A*'s property, and a bottle of wine, which was *B*'s. A stranger, who happened to come up at the time, was invited to partake of their fare, which was just sufficient for three persons; and at parting, being pleased with their behaviour, he gave them what Spanish money he had about him, which amounted to 6*s.* $5\frac{1}{4}$*d.*, to be equitably shared between them. Now as many shillings as a loaf cost pence would, with fourpence more, at the next town have bought six such loaves and four bottles of the same wine; and when the money was divided, *B* received 1*s.* $10\frac{1}{2}$*d.* more than *A*. What was the price of each loaf, and a bottle of wine?

Ans. A loaf cost 9 pence; and a bottle of wine $11\frac{1}{4}$ pence.

VIII. *Problems producing pure Equations.*

1. Find two numbers, which are in the proportion of 8 to 5, and whose product is equal to 360.

Ans. ± 24, and ± 15.

2. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128. What are the numbers?

Ans. $\pm$ 18, and $\pm$ 14.

3. In a court there are two square grass-plots; a side of one of which is 10 yards longer than the side of the other; and their areas are as 25 to 9. What are the lengths of the sides?

Ans. 25, and 15 yards.

4. A person bought two pieces of linen, which together measured 36 yards. Each of them cost as many shillings *per* yard, as there were yards in the piece; and their whole prices were in the proportion of 4 to 1. What were the lengths of the pieces?

Ans. 24, and 12 yards.

5. There are two numbers, whose sum is to the less as 5 to 2; and whose difference, multiplied by the difference of their squares, is 135. Required the numbers.

Ans. 9, and 6.

6. There are two numbers, which are in the proportion of 3 to 2; the difference of whose fourth powers is to the sum of their cubes as 26 to 7. Required the numbers.

Ans. 6, and 4.

7. There is a field in the form of a rectangular parallelogram, whose length is to its breadth in the proportion of 6 to 5. A part of this, equal to one-sixth of the whole, being planted, there remain for ploughing 625 square yards. What are the dimensions of the field?

Ans. The sides are 30, and 25 yards.

8. Some gentlemen made an excursion; and every one took the same sum. Each gentleman had as many servants attending him as there were gentlemen; and the number of pounds which each had was double the number of all the servants; and the whole sum of money taken out was £.3456. How many gentlemen were there?

Ans. 12.

9. Divide the number 49 into two such parts, that the quotient of the greater divided by the less may be to the quotient of the less divided by the greater as $\frac{4}{3}$ to $\frac{3}{4}$.

Ans. 28, and 21.

10. A detachment of soldiers from a regiment being ordered to march on a particular service, each company furnished four times as many men as there were companies in the regiment; but these being found to be insufficient, each company furnished 3 more men; when their number was found to be increased in the ratio of 17 to 16. How many companies were there in the regiment?

Ans. 12.

11. A charitable person distributed a certain sum amongst some poor men and women, the numbers of whom were in the proportion of 4 to 5. Each man received one-third of as many shillings as there were persons relieved; and each woman received twice as many shillings as there were women more than men. Now the men received all together 18*s.* more than the women. How many were there of each?

Ans. 12 men, and 15 women.

12. A gentleman who had a certain number of horses, kept part of them at livery stables, for which he paid £.4. 10*s*. *per* week. The rest he kept at home, and their number was to the number kept at the livery stables as 7 to 3. He found that the expence of keeping 5 at home was just equal to that of keeping 4 at the stables; and the number of shillings that 1 horse cost him at home was to the number of horses kept at home as 6 to 7. How many horses had he?

Ans. 6 at the livery stables, and 14 at home.

13. A city barge, with chairs for the company and benches for the rowers, went a summer excursion, with two bargemen on every bench. The number of gentlemen on board was equal to the square of the number of bargemen, and the number of ladies was equal to the number of gentlemen, twice the number of bargemen, and one over. Among other provisions, there were a number of turtles equal to the square root of the number of ladies; and a number of bottles of wine less than the cube of the number of turtles by 361. The turtles in dressing consumed a great quantity of wine, and the party having staid out till the turtles were all eaten, and the wine all gone, it was computed, that supposing them all to have consumed an equal quantity, (viz. gentlemen, ladies, bargemen, and turtles,) each individual would have consumed as many bottles as there were benches in the barge. Required the number of turtles.

Ans. 19.

14. From two towns, *C* and *D*, two travellers, *A* and *B*, set out to meet each other; and it appeared that when they met, *B* had gone 35 miles more than $\frac{3}{5}$ of the dis-

tance that *A* had travelled; but from their rate of travelling, *A* expects to reach *C* in 20 hours, and 50 minutes; and *B* to reach *D* in 30 hours. Required the distance of *C* from *D*.

Ans. 275 miles.

15. A farmer bought two flocks of sheep, the first of which contained 18 fewer than the second. If he had given for the first flock as many pounds as there were sheep in the second, and for the second as many pounds as there were sheep in the first, then the price of 6 sheep of the first flock would have been to the price of 7 sheep of the second in the proportion of 7 to 6. Required the numbers in each flock.

Ans. 108, and 126.

16. A poulterer bought a number of ducks and turkeys, the number of ducks exceeding the number of turkeys by 8. For each duck he gave half as many shillings as there were turkeys, and for each turkey half as many shillings as there were ducks. He afterwards bought another small flock of turkeys containing 4 fewer than the number of turkeys he bought before; and having given for each of them as many shillings as there were turkeys in the flock, he found, that if his former purchase had cost 16 shillings more, it would have cost exactly four times as much as the present one. How many ducks and turkeys did he buy at first?

Ans. 12 turkeys, and 20 ducks.

17. Two men, *A* and *B*, enter into partnership with stocks which are in the proportion of 9 to 8; and after trading one year, *A* finds his share of their gain to amount to one-third of his stock. They continued to trade for as many years as are equal to three-fourths of the number of pounds which *B* contributes to the stock, and find

their whole gain amount to £.1666. What did each contribute to the stock; and how many years did they trade?

Ans. *A* contributed £.63, *B* £.56; and the number of years is 42.

18. A person wishing to ascertain the area of a certain quadrilateral field, found that he could determine it the most readily by dividing it into two portions, one of which was of the form of a rectangular parallelogram, the shorter side of which measured 60 yards. The other was of the form of a right-angled triangle, whose shortest side was equal to the shorter side of the parallelogram, and the other side, containing the right-angle, was equal to the diagonal of the parallelogram; and the area of the triangle was to the area of the parallelogram as 5 to 8. What was the area of the field?

Ans. 7800 square yards.

19. A merchant laid out a certain sum upon a speculation, and found at the end of a year that he had gained £.69. This he added to his stock, and at the end of another year found that he had gained exactly as much *per* cent. as in the year preceding. Proceding in the same manner, and each year adding to his stock the gain of the year preceding, he found at the beginning of the fifth year that his stock was to the original stock as 81 to 16. What was the sum he first laid out?

Ans. £.138.

20. There is a number consisting of two digits, which being multiplied by the digit on the left hand, the product is 46; but if the sum of the digits be multiplied by the same digit, the product is only 10. Required the number.

Ans. 23.

21. From two towns, *C* and *D*, which were at the distance of 396 miles, two persons, *A* and *B*, set out at the same time, and meet each other, after travelling as many days as are equal to the difference of the number of miles they travelled *per* day; when it appears that *A* has travelled 216 miles. How many miles did each travel *per* day?

Ans. *A* went 36, and *B* 30.

22. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 is to the greater. What are the numbers?

Ans. 36, and 24.

23. It is required to find two numbers such, that the product of the greater and the cube of the less may be to the product of the less and cube of the greater as 4 to 9; and the sum of the cubes of the numbers may be 35.

Ans. 3, and 2.

24. The paving of two square court-yards cost £.205; a yard of each costing one-fourth of as many shillings as there were yards in a side of the other. And a side of the greater and less together measure 41 yards. Required the length of a side of each.

Ans. 25, and 16 yards.

25. A person bought a number of apples and pears, amounting together to 80. Now the apples cost twice as much as the pears: but had he bought as many apples as he did pears, and as many pears as he did apples, his apples would have cost 10*d.*, and his pears 3*s.* 9*d.* How many did he buy of each?

Ans. 60 apples, and 20 pears.

26. A person exchanged a quantity of brandy for a

quantity of rum and £.11. 5*s.*; the brandy and rum being each valued at as many shillings *per* gallon as there were gallons of that liquor. Now had the rum been worth as many shillings *per* gallon as the brandy was, the whole value of the rum and brandy would have been £.56. 5*s.* How many gallons were there of each?

Ans. 25 gallons of brandy, and 20 of rum.

27. There are two rectangular vats, the greater of which contains 20 solid feet more than the other. Their capacities are in the ratio of 4 to 5; and their bases are squares, a side of each of which is equal to the depth of the other. What are the depths?

Ans. 5 feet, and 4 feet.

28. Bought two square carpets for £.62. 1*s.*; for each of which I paid as many shillings *per* yard as there were yards in its side. Now had each of them cost as many shillings *per* yard as there were yards in a side of the other, I should have paid 17*s.* less. What was the size of each?

Ans. One contained 81, and the other 64 square yards.

29. The number of men in both fronts of two columns of troops, *A* and *B*, when each consisted of as many ranks as it had men in front, was 84; but when the columns changed ground, and *A* was drawn up with the front *B* had, and *B* with the front *A* had, the number of ranks in both columns was 91. Required the number of men in each column.

Ans. 2404, and 1296.

30. A field in the form of a rectangular parallelogram was planted with trees placed at such distances as to have four on every square yard. The expence of planting

was such, that every 40 trees cost one-third of as many shillings as there were yards in the diagonal of the parallelogram. But had they been planted at such a price as that every hundred should have cost as many shillings as there were yards in the shorter side of the parallelogram, the expence would have been less by £.224. Now a square described upon the diagonal of the parallelogram would be equal to $\frac{8}{3}$ of the square described on the less side, together with the square described on a line which is equal to the difference of the sides. Required the dimensions of the parallelogram.

Ans. The longer side is 80, and the shorter 60 yards.

IX. *Problems producing Adfected Quadratics.*

1. What two numbers are those, whose sum is 19, and whose difference multiplied by the greater is 60?

Ans. 12, and 7.

2. If the square of a certain number be taken from 40, and the square root of this difference be increased by 10, and the sum multiplied by 2, and the product divided by the number itself, the quotient will be 4. Required the number.

Ans. 6.

3. There is a field in the form of a rectangular parallelogram, whose length exceeds the breadth by 16 yards; and it contains 960 square yards. Required the length and breadth.

Ans. 40, and 24 yards.

4. A person being asked his age answered, if you add the square root of it to half of it, and subtract 12, there will remain nothing. Required his age.

Ans. 16.

5. Two casks of ale were bought for £.2. 18*s.* one of which contained 5 gallons more than the other, and the price *per* gallon was 2 shillings less than one-third of the number of gallons in the less. Required the number of gallons in each, and the price *per* gallon.

Ans. The numbers were 12, and 17, and the price *per* gallon 2 shillings.

6. From two places at the distance of 320 miles, two persons, *A* and *B*, set out at the same time to meet each other. *A* travelled 8 miles a day more than *B*, and the number of days in which they met was equal to half the number of miles *B* went in a day. How many miles did each travel *per* day, and how far did each travel?

Ans. *A* went 24, and *B* 16 miles *per* day; *A* went 128, and *B* 192 miles.

7. The difference between the hypothenuse and base of a right-angled triangle is = 6, and the difference between the hypothenuse and the perpendicular is = 3. What are the sides?

Ans. 9, 3, and 6.

8. In a parcel which contains 24 coins of silver and copper, each silver coin is worth as many pence as there are copper coins, and each copper coin is worth as many pence as there are silver coins, and the whole is worth 18 shillings. How many are there of each?

Ans. 6 of one, and 18 of the other.

9. A farmer received £.7. 4*s*. for a certain quantity of wheat, and an equal sum at a price less by 1*s*. 6*d*. *per* bushel for a quantity of barley, which exeeeded the quantity of wheat by 16 bushels: How many bushels were there of each?

Ans. 32 bushels of wheat, and 48 of barley.

10. Two messengers, *A* and *B*, were dispatched at the same time to a place 90 miles distant; the former of whom riding one mile an hour more than the other, arrived at the end of his journey an hour before him. At what rate did each travel *per* hour?

Ans. *A* went 10, and *B* 9 miles *per* hour.

11. Bought a number of books, consisting of folios, quartos, and octavos, for £.96. 12*s*. Fourteen folios (which was the whole number) cost 3 times as much as all the quartos; and one quarto cost as many shillings as there were quartos. The number of octavos was 32, and their value was such, that 4 of them cost as much as one quarto. Required the value of each, and the number of quartos.

Ans. There were 21 quartos, each folio cost $4\frac{1}{2}$ guineas, each quarto one guinea, and each octavo 5

12. A man travelled 105 miles, and then found that if he had not travelled so fast by 2 miles an hour, he should have been 6 hours longer in performing the same journey. How many miles did he go *per* hour?

Ans. 7 miles.

13. Bought two flocks of sheep for £.65. 13*s*., one, containing 5 more than the other. Each sheep cost as many shillings as there were sheep in the flock. Required the numbers in each flock.

Ans. 23, and 28.

14. A regiment of soldiers, consisting of 1066 men, is formed into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares?

Ans. 21, and 25.

15. What number is that, to which if 24 be added, and the square root of the sum extracted, this root shall be less than the original quantity by 18.

Ans. 25.

16. After taking the kings, queens, and knaves out of a pack of cards, the rest were divided into three heaps. The number of pips contained in the second heap was found to be 4 times the square of the number in the first heap; and had the third heap contained 5 more pips than it did, the number in it would have been exactly half of what the first and second heap contained. Required the number of pips in each heap.

Ans. 6, 144, and 70.

17. A tailor bought a piece of cloth for £.147, from which he cut off 12 yards for his own use, and sold the remainder for £.120. 5*s.* gaining 5 shillings *per* yard. How many yards were there, and what did it cost him *per* yard?

Ans. 49 yards, at £.3 *per* yard.

18. A regiment of foot was ordered to send 216 men on garrison duty, each company being to furnish an equal number; but before the detachment marched, 3 of the companies were sent on another service, when it was found that each company that remained was obliged to furnish 12 additional men, in order to make up the complement 216. How many companies were there in the

regiment, and what number of men was each company ordered to send at first?

Ans. There were 9 companies; and each was to send 24 men.

19. A poulterer bought 15 ducks and 12 turkeys for five guineas. He had two ducks more for 18 shillings than he had of turkeys for 20 shillings. What was the price of each?

Ans. The price of a duck was 3*s.*, and of a turkey 5*s.*

20. Two men, *A* and *B*, entered into a speculation, to which *B* subscribed £.15 more than *A*. After 4 months, *C* was admitted, who added £.50 to the stock; and at the end of 12 months from *C*'s admission they found they had gained £.159; when *A* withdrawing received for principal and gain £.88. What did he originally subscribe?

Ans. £.40.

21. A wall was built round a rectangular court to a certain height. Now the length of one side of the court was two yards less than 8 times the height of the wall, and the length of the adjacent side was 5 yards less than 6 times the height of the wall; and the number of square yards in the court was greater than the number in the wall by 178. Required the dimensions of the court, and the height of the wall.

Ans. The sides were 30, and 19, and the height 4 yards.

22. A ship containing 74 sailors, and a certain number of soldiers besides officers, took a prize. The sailors received each one-third as many pounds as there were soldiers, and the soldiers received £.3 a piece less, and

£.768 fell to the share of the officers. Had the officers however received nothing, the soldiers and sailors might have received half as many pounds *per* man, as there were soldiers. How many soldiers were there, and how much did each receive?

Ans. There were 36 soldiers, each soldier received £.9, and each sailor £.12.

23. A poulterer going to market to buy turkeys, met with four flocks. In the second were 6 more than three times the square root of double the number in the first. The third contained three times as many as the first and second; and the fourth contained 6 more than the square of one-third of the number in the third; and the whole number was 1938. How many were there in each flock?

Ans. The numbers were 18, 24, 126, 1770, respectively.

24. A body of men are just sufficient to form a hollow equilateral wedge, three deep; and if 597 be taken away, the remainder will form a hollow square, four deep, the front of which contains one man more than the square root of the number contained in a front of the wedge. What is the number of men?

Ans. 693.

25. Two men, *A* and *B*, undertake to perform a piece of work in four days, for which they are to receive a certain number of shillings; but after some time, finding that they shall not be able to finish it in the time proposed, they call in *C* to assist them, and upon an equitable division of the money, *C* receives a sum equal to the square root of the whole number of shillings; but had they been obliged to call in *C* to their assistance

$1 + \frac{1}{9}$ of a day sooner, his share of the money would have been two-fifths more. How long did *C* work, and what did he receive?

Ans. He worked 2 days, and received 5 shillings.

26. A cask, whose content is 20 gallons, is filled with brandy, a certain quantity of which is then drawn off into another cask of equal size; this last cask is then filled with water; after which the first cask is filled with the mixture, and it appears, that if $6\frac{2}{3}$ gallons of the mixture be drawn off from the first into the second cask, there will be equal quantities of brandy in each. Required the quantity of brandy first drawn off.

Ans. 10 gallons.

27. There are three numbers, the difference of whose differences is 5; their sum is 20; and their continual product 130. Required the numbers.

Ans. 2, 5, and 13.

28. There are three numbers, the difference of whose differences is 3; their sum is 21; and the sum of the squares of the greatest and least is 137. Required the numbers.

Ans. 4, 6, 11.

29. There is a number consisting of 2 digits, which when divided by the sum of its digits gives a quotient greater by 2 than the first digit. But if the digits be inverted, and then divided by a number greater by unity than the sum of the digits, the quotient is greater by 2 than the preceding quotient. Required the number.

Ans. 24.

30. A certain sum was to be raised on three estates belonging to *A*, *B*, and *C*, at the rate of one shilling *per* acre. Now the number of acres *A* and *B* had, were as

3 to 7; and if the number of acres in the whole were divided by one-third of the product of the numbers in the first and third, the quotient would be $\frac{3}{4}$. Also the sum paid by *A* and *C* was 36 shillings less than the sum of three times the money paid by *C*, and $\frac{2}{7}$ of the money paid by *B*. Of how many acres did each estate consist; and what was the whole sum to be raised?

> *Ans.* *A* had 12, *B* 28, and *C* 30 acres; and the sum was £.3. 10*s*.

31. A butcher bought a certain number of calves and sheep, and for each of the former gave as many shillings as there were sheep, and for each of the latter one-fourth as much. Now had he given 4 shillings more for each of the former, and 2 shillings more for each of the latter, he would have paid seven pounds more. But had a sheep cost as much as a calf, he would have expended £.56. 8*s*. How many did he buy of each; and what were their prices?

> *Ans.* 23 calves, and 24 sheep; and their prices were 24, and 6 shillings, respectively.

32. Two persons, *A* and *B*, comparing their wages, observe that if *A* had received *per* day in addition to what he does receive, a sum equal to one-fourth of what *B* received *per* week, and had worked as many days as *B* received shillings *per* day, he would have received £.2. 8*s*.; and had *B* received 2 shillings a day more than *A* did, and worked for a number of days equal to half the number of shillings he received *per* week, he would have received £.4. 18*s*. What were their daily wages?

> *Ans.* *A*'s 5 shillings, and *B*'s 4.

33. There are three towns in the order of the letters, *A*, *B*, *C*. The difference betwen the distances from *A* to *B* and from *B* to *C* is greater by four miles than the distance from *B* to *D*. Also the number of miles between *B* and *D* is equal to two-thirds of the number between *A* and *C*. And the number between *A* and *B* is to the number between *C* and *D* as seven times the number between *B* and *C* : 26. Required the respective distances.

Ans. $AB = 42$, $BC = 6$, $CD = 26$ miles.

34. A person bought a quantity of cloth of two sorts for £.7. 18*s*. For every yard of the better sort he gave as many shillings as he had yards in all; and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards had he of each?

Ans. 7 yards of the better, and 9 of the worse.

35. A farmer sold a certain number of bushels of barley, and ten bushels of wheat for £.7. 19*s*. Now each bushel of wheat cost within 3 shillings as much as two bushels of barley. He afterwards sold as many bushels of barley and four more, and fifteen bushels of wheat, and received two shillings *per* bushel more for his wheat and barley than he did before; when he found that if he had received £.1. 4*s*. more, he should just have received twice as much as he did before. How many bushels of barley did he sell the first time; and what were the prices *per* bushel of the wheat and barley?

Ans. 7 bushels of barley; and the prices of wheat and barley were 11*s*. and 7*s*. *per* bushel.

36. In digging among some ruins the workmen found 9 urns, together containing 60 gold coins; the second

and eighth containing 8 and 4 respectively. They secreted a certain number of these, greater than the number they left; which being afterwards recovered, it was found that the number of urns secreted was to the number left as the number of coins secreted was to the number remaining. Now if instead of taking the second urn they had carried off the eighth, then the number of coins taken away would have been to the number remaining as the square of the number of urns secreted to the difference between that square and 20 times the number of urns remaining. Required the numbers of urns and coins secreted.

Ans. 6 urns, and 40 coins.

37. Two men, *A* and *B*, set out from the same place to travel. *A* goes in 6 days twice as many miles as *B* goes in 5 days, but does not arrive at the end of his journey till 5 days after *B* has arrived at the end of his, when he finds that he has travelled 259 miles more than *B*. But had *B* gone 2 miles *per* day more than he did, and *A* stopped 6 days sooner, *A* would then have gone only 37 miles more than *B*. How many miles did each travel *per* day, and how many days did they travel?

Ans. *A* travelled 11 days, and 35 miles *per* day,
B travelled 6 days, and 21 miles *per* day.

38. Bacchus caught Silenus asleep by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask. After that Silenus awoke, and drank what Bacchus had left. Had they drunk both together, it would have been emptied two hours sooner, and Bacchus would have drunk only half what he left Silenus. Required the time in which they would empty the cask separately.

Ans. Silenus in 3 hours, and Bacchus in 6.

39. Two persons, *A* and *B*, comparing the distances they have travelled, found that the square of the number of miles which *A* usually walked *per* hour, exceeded the square of the number which *B* usually walked by 5; and that if to the square of the product of those numbers there be added the square of the sum of their fourth powers, augmented by the product of the square of the difference of their squares into the square of the product of the numbers themselves, the aggregate amount would be 10345. How many miles did each walk *per* hour?

Ans. *A* walked 3, and *B* 2 miles.

40. Two plantations, one of an oblong, and the other of a square form, contain the same number of trees, and they have one fence common to both, viz. that which bounds the end of the oblong one. Upon every pole in the square are planted as many trees as there are poles in the square, and upon every pole in the oblong four times as many trees as there are poles in the breadth, besides 144 in the hedges. Also the area of the oblong wants 6 poles to be to the area of the square as 3 to 2. Required the number of trees.

Ans. 1296.

41. There are two sorts of metal, each being a mixture of gold and silver, but in different proportions. Two coins from these metals of the same weight are to each other in value as 11 to 17; but if to the same quantities of silver as before in each mixture double the former quantities of gold had been added, the values of two coins from them of equal weights would have been to each other as 7 to 11. Determine the proportion of gold to silver in each mixture, the values of equal weights of gold and silver being as 13 to 1.

Ans. The proportion of gold to silver is 1 : 9 in the first mixture, and 1 : 4 in the second.

42. A mason has two cubical pieces of white marble of exactly the same size, and two cubical equal pieces of black, larger than the other. The number of solid yards in the four pieces is 9 more than 11 times the number of yards in a side of a white one, together with 12 times the number in a side of a black one. He afterwards finds another block, the length of which is 2 yards longer than a side of one of the white pieces, and the width 4 times the length of a side of the black one; and this when laid on its largest side occupies a space greater by 3 yards than the difference between 4 times the space occupied by a black, and 3 times the space occupied by a white one. Required the dimensions of the blocks.

Ans. The side of a white block is 1 yard, and of a black one 3 yards; the length of the other is 3 yards, and the width 12 yards.

X. *Problems in Arithmetical and Geometrical Progressions.*

1. There are three numbers in arithmetical progression, whose sum is 21; and the sum of the first and second is to the sum of the second and third as 3 to 4. Required the numbers.

Ans. 5, 7, 9.

2. There are six towns in the order of the letters, *A*, *B*, *C*, *D*, *E*, *F*, whose distances from each other are in an increasing arithmetical progression. The distance from *A* to *C* is 16 miles, and from *C* to *E* is 24 miles. Required their respective distances.

Ans. From *A* to *B* is 7, from *B* to *C* 9, from *C* to *D* 11, from *D* to *E* 13, and from *E* to *F* 15 miles.

3. A person makes a mixture of 51 gallons, consisting of brandy, rum, and water, the quantities of which are in arithmetical progression. The number of gallons of brandy and rum together is to the number of gallons of rum and water together as 8 to 9. Required the quantities of each.

Ans. 15 gallons of brandy, 17 of rum, and 19 of water.

4. During a scarcity, a person wished to make a mixture of 24 bushels, consisting of wheat, oats, and barley, the quantities of each forming an increasing arithmetical progression. Not being able however to procure any barley, he mixed additional quantities of wheat and oats in the proportion of 2 to 3, so as to complete his 24 bushels, when he found the whole quantities of wheat and oats to be in the proportion of 5 to 7. How many bushels of each did he originally intend to mix?

Ans. 6 of wheat, 8 of oats, and 10 of barley.

5. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers? *Ans.* 54, 18, 6, and 2.

6. A person employed three workmen, whose daily wages were in arithmetical progression. The number of days they worked was equal to the number of shillings that the second received *per* day. The whole amount of their wages was seven guineas, and the best workman received 28 shillings more than the worst. What were their daily wages?

Ans. 5, 7, and 9 shillings.

7. There are three numbers in geometrical progression; the sum of the first and second of which is 9, and the sum of the first and third is 15. Required the numbers.

Ans. 3, 6, 9.

8. There are three numbers in geometrical progression, whose sum is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

Ans. 2, 4, 8.

9. There are three numbers in geometrical progression, whose continued product is 64, and the sum of their cubes is 584. Required the numbers.

Ans. 2, 4, 8.

10. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. Required the numbers.

Ans. 1, 3, 9, 27.

11. From two towns which were 168 miles distant, two persons, *A* and *B*, set out to meet each other; *A* went 3 miles the first day, 5 the next, 7 the third, and so on; *B* went 4 miles the first day, 6 the next, and so on. In how many days did they meet?

Ans. 8.

12. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he has been gone 3 days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first?

Ans. In 2, and 9 days.

13. A person has two pieces of ground, one of which is in the form of an equilateral triangle, and the

other of a rectangular parallelogram, one side of which is equal to a side of the triangle, and the other side is 8 yards less. These he plants with trees at the distance of two yards from each other, and finds that there are 5 more on the rectangle than on the triangle. What are the lengths of the sides?

Ans. A side of the triangle is 20 yards, and the sides of the parallelogram are 20 and 12 yards.

14. There are four numbers in arithmetical progression, whose sum is 28; and their continual product is 585. Required the numbers.

Ans. 1, 5, 9, 13.

15. There are four numbers in arithmetical progression; the sum of the squares of the first and second is 34; and the sum of the squares of the third and fourth is 130. Required the numbers.

Ans. 3, 5, 7, 9.

16. The sum of £.700 was divided among four persons, whose shares were in geometrical progression; and the difference between the greatest and least was to the difference between the means as 37 to 12. What were their respective shares?

Ans. £.108, £.144, £.192, £.256.

17. Five persons undertake to reap a field of 87 acres. The five terms of an arithmetical progression, whose sum is 20, will express the times in which they can severally reap an acre; and they all together can finish the undertaking in 60 days. In how many days can each separately reap an acre.

Ans. 2, 3, 4, 5, 6 days.

18. Out of a vessel containing 24 gallons of pure spirit, a vintner drew off at three successive times a certain number of gallons, which formed an increasing arithmetical progression, in which the difference between the squares of the extremes was equal to 16 times the mean, and filled up the vessel with water after each draught, till he found what he last drew off, reduced it to one-sixth of its original strength. Required the number of gallons of pure spirit drawn off each time.

Ans. 12, 8, $3\frac{1}{3}$.

19. The Fly starts 10 miles before the Telegraph; but the Fly coachman having made an appointment with the driver of the Telegraph, walks his horses so as to be overtaken at the end of the second mile. Now it is observed, that the number of revolutions made in a given time by the hinder wheel of the Fly, its fore wheel, and the hinder wheel of the Telegraph increase in arithmetical progression, and that the circumferences of these wheels, viz. of the fore wheel of the Fly, its hinder wheel, and the hinder wheel of the Telegraph increase in a geometrical progression, whose common ratio is the same as the common difference of the arithmetical progression. It is required to find the ratio that the wheels bear to each other.

Ans. 1, 2, 4 are their proportional lengths.

20. A company of merchants fitted out a privateer, each subscribing £.100. The captain subscribed nothing, but was entitled to a £.100 share at the end of every certain number of months. In the course of 25 months he captured three prizes, which were in geometrical progression, the middle term being one-fourth of the cost of the equipment, the common ratio

the number of months which entitled the captain to his £.100 share, and their sum £.1375 more than the cost of the equipment. After deducting £.875 for prize money to the crew, the captain's share of the remainder amounted to one-fifth of that of the company. Required the number of merchants, and the captain's pay.

Ans. The number of merchants was 25, and the captain was entitled to a £.100 share at the end of every 5 months.